Brief summary of functional analysis
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Stephen Becker, stephen.becker@colorado.edu

Standard theorems. When necessary, I used Royden's and Kreyzsig's books as a reference. Version 5, 10/27/17

Contents

1 Fundamental theorems

Theorem 1 (Hahn-Banach)**.** *See our text for details. Proof requires Zorn's lemma/Axiom of choice*

A subset $Y \subset X$ is *nowhere dense* if any open set $U \subset X$ contains a ball $B \subset U \subset X$ such that $B \cap Y = \emptyset$. That is, Y is nowhere dense if the interior of the closure of Y is empty. For example, the Cantor set is a nowhere dense subset of the unit interval.

Theorem 2 (Baire Category Theorem)**.** *A complete metric space cannot be written as the countable union of nowhere dense sets.*

Theorem 3 (Uniform Boundedness Theorem, aka Banach Steinhaus Theorem (Thm. 8.39 in our text))**.** *Let $T_n : X \to Y$ be a sequence of bounded linear operators from a Banach space X into a normed linear space Y. Assume that for each $x \in X$, there is a real number c_x such that*

$$\|T_n x\| \le c_x, \quad \forall\, n = 1, 2, \cdots,$$

Then there is a real number c such that $\|T_n\| \le c, \quad \forall\, n = 1, 2, \cdots.$, i.e., $\sup_{n\in\mathbb{N}} \|T_n\| \le c$.

This follows from the Baire category theorem.

An **open mapping** is one that maps open sets to open sets.

Theorem 4 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces. Then any bounded linear operator T from X onto Y (that is, T is surjective) is an open mapping. Consequently, if T is bijective, then T^{-1} is continuous and hence bounded (as well as linear).*

This follows from the uniform boundedness theorem.

Theorem 5 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and let $D \subset X$ be a subspace. Let $T : D \to Y$ be a closed linear operator. If D is closed in X, then T is bounded.*

This follows from the uniform boundedness theorem. We can also state it like this:

Theorem 6 (Closed Graph Theorem, variant)**.** *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator. The graph of T is a subset of $X \otimes Y$ defined by*

$$\mathrm{graph}(T) = \{(x, T(x)) \mid x \in X\}.$$

Then graph(T) *is a closed subspace iff T is bounded.*

Theorem 7 (See Thm. 6.29 in our text)**.** *Every Hilbert space has an orthonormal basis. Uses Zorn's lemma in the proof.*

Theorem 8 (Banach-Alaoglu)**.** *See Thm. 5.61, 8.45 in our text. Let X be a normed linear space, then the closed unit ball B^* of its dual space X^* is compact with respect to the weak-$*$ topology.*

2 Hahn-Banach in more detail

Our book doesn't have the most general version, so here is a more general version, but we don't prove it (proof relies on Zorn's lemma). Royden and Reed/Simon have proofs, for example.

Theorem 9. *(Hahn-Banach Theorem) Let X be a linear space over a field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). Let $p : X \to \mathbb{R}$ be a real-valued functional on X satisfying*

$$\begin{aligned} p(x+y) &\leq p(x) + p(y), \quad \forall\, x, y \in X \quad \text{“sub-linear”} \\ p(\alpha x) &= |\alpha|\, p(x), \quad \forall\, \alpha \in \mathbb{F},\ x \in X \quad \text{“positive homogeneous”.} \end{aligned}$$

Furthermore, let $Z \subset X$ be a subspace of X and let $f : Z \to \mathbb{F}$ be a linear functional on Z such that

$$|f(x)| \leq p(x), \quad \forall\, x \in Z.$$

Then f has a linear extension $\tilde{f} : X \to \mathbb{F}$ with

$$|\tilde{f}(x)| \leq p(x), \quad \forall\, x \in X.$$

Note that sub-linearity implies $p(x) = 0$, and using this with the positive homogeneous property implies $p(x) \geq 0$ for all $x \in X$.

3 Proof of the Baire category theorem

Let X be a metric space and $M \subset X$ be a subset. A point $x \in M$ is called an interior point of M if there is $\epsilon > 0$ such that $B_\epsilon(x) \subset M$. We will begin with the following definition regarding metric spaces.

Definition 10. *Let X be a metric space and $M \subset X$ be a subset. Then M is said to be*

1. ***rare*** *(or* ***nowhere dense****) in X if its closure $\overline{M}$ has no interior point (e.g., $\mathbb{Z}$ in $\mathbb{R}$)*
2. ***meager*** *(or of the first category) in X if M is the union of countably many sets each of which is rare in X.*
3. *nonmeager (or of the second category) in X if M is not meager in X.*

Royden uses the term **hollow** for a subset with empty interior. A set is hollow iff its complement is dense. A set is nowhere dense if its closure is hollow.

Now we will state and proof the important Baire's Category Theorem. The proof can be shortened lightly if we used the Cantor Intersection Theorem (cf. Royden). The following proof is from Kreyszig.

Theorem 11. *(Baire's Category Theorem) If a metric space $X \neq \emptyset$ is complete, then it is nonmeager in itself. Consequently, if $X \neq \emptyset$ is complete and*

$$X = \bigcup_{k=1}^{\infty} A_k, \quad A_k \text{ closed}$$

then at least one A_k contains a nonempty open subset of X.

Royden's version is that if X is complete, and $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$ is a collection of open dense subsets of X, then the intersection $\cap_{n=1}^{\infty} \mathcal{O}_n$ is also dense; the proof of this is a homework problem for us (so do not use this exact version of the theorem on the HW, but you can use the other variants of Baire). Most compactly stated, an open subset of a complete space is of the second category.

Proof. Assume, on the contrary, that X is meager in itself. By definition,

$$X = \bigcup_{k=1}^{\infty} M_k,$$

where each M_k is rare in X. Since M_1 is rare in X, we have $\overline{M}_1 \neq X$ since X itself is open and therefore contains interior points. So, the complement $(\overline{M}_1)^c$ is nonempty and open. Thus, there is a point $p_1 \in (\overline{M}_1)^c$ and $0 < \epsilon_1 < \frac{1}{2}$ such that

$$B_1 \stackrel{\text{def}}{=} B_{\epsilon_1}(p_1) \subset (\overline{M}_1)^c.$$

Note that $B_1 \cap M_1 = \emptyset$ and $\epsilon_1 < 2^{-1}$.

Since M_2 is rare, its closure $\overline{M}_2$ does not contain a nonempty open set. In particular, $\overline{M}_2$ does not contain $B_{\frac{1}{2}\epsilon_1}(p_1)$ (though they could overlap). So, the set $(\overline{M}_2)^c \cap B_{\frac{1}{2}\epsilon_1}(p_1)$ is nonempty and open. Thus, there is a point[1] $p_2 \in (\overline{M}_2)^c \cap B_{\frac{1}{2}\epsilon_1}(p_1)$ and $0 < \epsilon_2 < \frac{1}{2}\epsilon_1$ such that

$$B_2 \stackrel{\text{def}}{=} B_{\epsilon_2}(p_2) \subset (\overline{M}_2)^c \cap B_{\frac{1}{2}\epsilon_1}(p_1).$$

Note that $B_2 \cap M_2 = \emptyset$, $B_2 \subset B_{\frac{1}{2}\epsilon_1}(p_1) \subset B_1$ and $\epsilon_2 < 2^{-2}$.

Continuing in this fashion, we obtain a sequence of balls B_k such that

$$B_k \stackrel{\text{def}}{=} B_{\epsilon_k}(p_k), \quad B_k \cap M_k = \emptyset, \quad B_{k+1} \subset B_{\frac{1}{2}\epsilon_k}(p_k) \subset B_k, \quad \epsilon_k < 2^{-k}.$$

Now we will show that the sequence $\{p_k\}$ is Cauchy. Given $\epsilon > 0$, there is N such that $2^{-N+1} < \epsilon$. For any $m, n > N$, we have $B_m \subset B_N$ and $B_n \subset B_N$, so

$$d(p_m, p_n) \leq d(p_m, p_N) + d(p_N, p_n) < \epsilon_N + \epsilon_N < 2^{-N} + 2^{-N} = 2^{-N+1} < \epsilon.$$

This proves $\{p_k\}$ is Cauchy.

Since X is complete, there is $p \in X$ such that $p_k \to p$. Fixed m. For any $n > m$, using $B_n \subset B_{\frac{1}{2}\epsilon_m}(p_m)$, we have

$$d(p_m, p) \leq d(p_m, p_n) + d(p_n, p) < \frac{1}{2}\epsilon_m + d(p_n, p).$$

Taking $n \to \infty$, we have

$$d(p_m, p) \leq \frac{1}{2}\epsilon_m < \epsilon_m.$$

Therefore, $p \in B_m$ for all m. Since $B_k \cap M_k = \emptyset$, we have $p \notin M_k$ for all k. Hence $p \notin \cup_{k=1}^{\infty} M_k = X$. This is a contradiction. □

3.1 Cantor intersection theorem

For reference, here is the Cantor intersection theorem, from Royden.

Theorem 12 (Cantor intersection theorem)**.** *Let X be a metric space. Then X is complete if and only if whenever $\{F_n\}_{n=1}^{\infty}$ is a contracting (nested) sequence of non-empty closed subsets of X, there is a point $x \in X$ for which $\cap_{n=1}^{\infty} F_n = \{x\}$.*

The sequence is nested in the sense that $F_{n+1} \subset F_n$. We say a sequence is **contracting** if the diameter of the sets goes to zero. From Royden, "a very rough geometric interpretation of the Cantor Intersection Theorem is that a metric space fails to be complete because it has 'holes'. " The proof is straightforward: if X is complete, then we can pick $x_n \in F_n$ and show (x_n) is Cauchy and its limit is in the intersection $\cap F_n$. Conversely, to prove X is complete, let (x_n) be Cauchy and define $F_n = \overline{\{x_k \mid k \geq n\}}$ and use this to show x_n converges to a limit.

The **topological Cantor intersection** theorem says: if X is a Hausdorff Topological space, and $F_{n+1} \subset F_n$, with each F_n non-empty and compact, then $\cap F_n$ is non-empty. The proof is via open-coverings (see wikipedia).

[1] This is where we use the axiom of choice

4 Proof of the Uniform Boundedness/Banach-Steinhaus theorem

Theorem 13 (Uniform Boundedness Theorem, aka Banach Steinhaus Theorem)**.** *Let $T_n : X \to Y$ be a sequence of bounded linear operators from a Banach space X into a normed linear space Y. Assume that for each $x \in X$, there is a real number c_x such that*

$$\|T_n x\| \le c_x, \quad \forall\, n = 1, 2, \cdots,$$

Then there is a real number c such that

$$\|T_n\| \le c, \quad \forall\, n = 1, 2, \cdots.$$

Proof. For each positive integer k, we define A_k as the set of all x such that

$$\|T_n x\| \le k, \quad \forall\, n = 1, 2, \cdots.$$

We will prove that A_k is closed. For each $x \in \overline{A}_k$, there is a sequence $\{x_j\} \subset A_k$ such that $x_j \to x$. Since $x_j \in A_k$, we have

$$\|T_n x_j\| \le k, \quad \forall\, n = 1, 2, \cdots.$$

Letting $j \to \infty$ and using the fact that T_n is continuous, we have $\|T_n x\| \le k$ for all n. So, $x \in A_k$. This shows that A_k is closed.

By assumption of the theorem, we have

$$X = \bigcup_{k=1}^{\infty} A_k.$$

Since X is complete, by the Baire's Category Theorem (Theorem 11), there is k_0 such that A_{k_0} contains an open ball, namely

$$B_0 \stackrel{\text{def}}{=} B_r(x_0) \subset A_{k_0}.$$

Let $x \in X$ be arbitrary with $x \ne 0$. Define a point z by

$$z = x_0 + \frac{r}{2\|x\|} x. \quad (x = \frac{2\|x\|}{r}(z - x_0).)$$

Then $\|z - x_0\| < r$. So, we have $z \in B_0$. Since $B_0 \subset A_{k_0}$, we have $\|T_n z\| \le k_0$, $\forall\, n$. In addition, since $x_0 \in B_0$, we have $\|T_n x_0\| \le k_0$, $\forall\, n$. Thus, for each n, we have

$$\|T_n x\| = \frac{2\|x\|}{r}\|T_n(z - x_0)\| \le \frac{2\|x\|}{r}(\|T_n z\| + \|T_n x_0\|) \le \frac{4\|x\|}{r} k_0.$$

Hence

$$\|T_n\| = \sup_{x \in X, x \ne 0} \frac{\|T_n x\|}{\|x\|} \le \frac{4}{r} k_0.$$

□

5 Open-mapping theorem in more detail

For references, see Kreyszig, Royden or Reed and Simon. We begin this section with the following notation. Let X be a linear space and let $A \subset X$ be a subset. We use the following notation.

$$\begin{aligned} \alpha A &= \{x \in X \mid x = \alpha\, a,\ a \in A\}, \\ A + w &= \{x \in X \mid x = a + w,\ a \in A\}, \end{aligned}$$

where $\alpha \in K$ and $w \in X$.

First we will prove the following lemma which relies on the Baire category theorem — this lemma is really the meat of the proof of the main theorem. We will make some use of the **translation principle** which states, roughly, that if we can control a linear operator T on a ball, then we can basically translate that ball to the origin due to the linearity of T. [2]

Lemma 14. *Let T be a bounded linear operator from a Banach space X onto a Banach space Y. Then the image of the open ball $B_0 = B_1(0) \subset X$, that is $T(B_0)$, contains an open ball with center 0 in Y.*

Proof.

The proof has three steps. We will prove

[2] Another version of the "translation principle" is that if $T : X \to Y$ is linear and X, Y Banach, then T is bounded if and only if the set $T^{-1}(B_1(0))$ has a non-empty interior, where $B_1(0) = \{y \in Y \mid \|y\| \le 1\}$ is the unit ball in Y. See Reed and Simon p. 80.

(a) $\overline{T(B_1)}$, where $B_1 = B_{2^{-1}}(0) \subset X$, contains an open ball B^*.

(b) $\overline{T(B_n)}$, where $B_n = B_{2^{-n}}(0) \subset X$, contains an open ball V_n with center 0.

(c) $T(B_0)$ contains an open ball with center 0.

Proof. (a) Clearly, we have $\cup_{k=1}^{\infty} kB_1 \subset X$. For any $x \in X$, there is k ($k > 2\|x\|$) such that $x \in kB_1$. So, $X \subset \cup_{k=1}^{\infty} kB_1$. Thus we have

$$X = \bigcup_{k=1}^{\infty} kB_1.$$

Since T is surjective,

$$Y = T(X) = T(\bigcup_{k=1}^{\infty} kB_1) = \bigcup_{k=1}^{\infty} T(kB_1).$$

Since T is linear, we have

$$Y = \bigcup_{k=1}^{\infty} T(kB_1) = \bigcup_{k=1}^{\infty} kT(B_1) = \bigcup_{k=1}^{\infty} \overline{kT(B_1)}.$$

Since Y is complete, by the Baire's category theorem (Theorem 11), there is k such that $\overline{kT(B_1)}$ contains an open ball. This implies that $\overline{T(B_1)}$ must also contain an open ball, namely, there is $B^* \stackrel{\text{def}}{=} B_\epsilon(y_0)$ such that $B^* \subset \overline{T(B_1)}$.

(b) We will first prove that $B_\epsilon(0) = B^* - y_0 \subset \overline{T(B_0)}$. Since $B^* \subset \overline{T(B_1)}$ by (a), we have $B^* - y_0 \subset \overline{T(B_1)} - y_0$. It suffices to prove

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}.$$

Let $y \in \overline{T(B_1)} - y_0$. Then $y + y_0 \in \overline{T(B_1)}$. Notice that $y_0 \in \overline{T(B_1)}$ since $B^* \subset \overline{T(B_1)}$. Then there are sequences $u_n = Tw_n \in T(B_1)$ and $v_n = Tz_n \in T(B_1)$ such that

$$u_n \to y + y_0, \qquad v_n \to y_0,$$

where $w_n, z_n \in B_1$. Observing that

$$\|w_n - z_n\| \le \|w_n\| + \|z_n\| < \frac{1}{2} + \frac{1}{2} = 1.$$

So, $w_n - z_n \in B_0$. Also,

$$T(w_n - z_n) = Tw_n - Tz_n = u_n - v_n \to y.$$

Hence, $y \in \overline{T(B_0)}$. This proves the following

$$B_\epsilon(0) = B^* - y_0 \subset \overline{T(B_0)}.$$

Let $B_n = B_{2^{-n}}(0)$. Since T is linear, we have $\overline{T(B_n)} = 2^{-n}\overline{T(B_0)}$. Let $V_n = B_{\epsilon\, 2^{-n}}(0)$. Then

$$V_n = 2^{-n} B_\epsilon(0) \subset 2^{-n}\overline{T(B_0)} = \overline{T(B_n)}.$$

This proves (b).

(c) Finally, we will prove that

$$V_1 = B_{\epsilon\, 2^{-1}}(0) \subset T(B_0).$$

Let $y \in V_1$. Since $V_1 \subset \overline{T(B_1)}$, there is $x_1 \in B_1$ such that

$$\|y - Tx_1\| < \frac{\epsilon}{4}.$$

Then we have $y - Tx_1 \in V_2$. Since $V_2 \subset \overline{T(B_2)}$, there is $x_2 \in B_2$ such that

$$\|y - Tx_1 - Tx_2\| < \frac{\epsilon}{8}.$$

Continuing in this fashion, we have, for each n, there are $x_n \in B_n$ such that

$$\|y - \sum_{k=1}^{n} Tx_k\| < \frac{\epsilon}{2^{n+1}}.$$

Let $z_n = x_1 + x_2 + \cdots + x_n$. The above inequality becomes

$$\|y - Tz_n\| < \frac{\epsilon}{2^{n+1}}, \quad \forall\, n.$$

Namely, $Tz_n \to y$. Since $x_k \in B_k$, we have $\|x_k\| < 2^{-k}$. So, for $n > m$,

$$\|z_n - z_m\| \leq \sum_{k=m+1}^{n} \|x_k\| < \sum_{k=m+1}^{\infty} \frac{1}{2^k} \to 0,$$

as $m \to \infty$. Thus, the sequence $\{z_n\}$ is Cauchy. Since X is complete, there is $x \in X$ such that $z_n \to x$ and $x = x_1 + x_2 + \cdots$. Notice that

$$\|x\| \leq \sum_{k=1}^{\infty} \|x_k\| < \frac{1}{2} + \sum_{k=2}^{\infty} \|x_k\| \leq \frac{1}{2} + \frac{1}{2} = 1.$$

So, $x \in B_0$. Since T is continuous, we have $Tz_n \to Tx$. Hence $y = Tx$. That is $y \in T(B_0)$.

□

We are now in a position to introduce open mapping.

Definition 15. *Let X and Y be metric spaces and let $D \subset X$ be a subspace. A mapping $T : D \to Y$ is called an **open mapping** if for every open set in D, the image is an open set in Y.*

Theorem 16 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces. Then any bounded linear operator T from X onto Y is an open mapping. Consequently, if T is bijective, then T^{-1} is continuous and hence bounded.*

Proof. Let $A \subset X$ be an arbitrary open subset of X. We will show that the image $T(A)$ is open in Y. That is, for any $y = Tx \in T(A)$, the set $T(A)$ contains an open ball centered at y.

Let $y \in T(A)$. Then $y = Tx$ with $x \in A$. Since A is open, there is $r > 0$ such that $B_r(x) \subset A$. Thus

$$B_1(0) \subset \frac{1}{r}(A - x).$$

By Lemma 14, the image $T(\frac{1}{r}(A - x))$ contains an open ball with center 0. That is, there is $\epsilon > 0$, such that

$$B_\epsilon(0) \subset T(\frac{1}{r}(A - x)).$$

Since T is linear, we have

$$B_\epsilon(0) \subset \frac{1}{r}(T(A) - Tx).$$

Since $y = Tx$, the above relation implies

$$B_{r\epsilon}(y) \subset T(A).$$

Hence $T(A)$ contains an open ball with center y. □

5.1 Closed graph theorem

We have some variants below (due to Kreyszig?), but Royden's version condenses it to its essentials. Both versions use the open mapping theorem to make for short proofs.

5.1.1 Royden's version

First, we use a corollary of the open mapping theorem:

Corollary 17. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a linear space X, and assume both $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach. Suppose there is a constant $c \geq 0$ for which*

$$\|\cdot\|_2 \leq c\|\cdot\|_1 \text{ on } X.$$

Then these two norms are equivalent.

Proof. The identity mapping is bijective, and bounded by the assumption of the corollary, therefore the inverse is bounded by the open mapping theorem. □

Definition 18. *A linear operator $T : X \to Y$ between normed linear spaces X and Y is said to be* **closed** *whenever for all $(x_n) \subset X$,*

$$(x_n) \to x \quad AND \quad T(x_n) \to y, \quad then \quad T(x) = y.$$

This is almost the definition of sequential continuity, except we have added the assumption that $T(x_n)$ does converge, so it is weaker. Hence we automatically have that *continuous functions are closed* (in a metric space).

The **graph** of a mapping is the set $\text{graph}(T) = \{(x, T(x))\}$. Thus an operator is closed (as an operator) if and only if its graph is closed (as a set) in the product space $X \times Y$ (more precisely, the graph is a closed *subspace*, since T is linear so its range is always a subspace). Specifically, T closed means that if $(x_n, T(x_n)) \to (x, y)$, then $(x, y) \in \text{graph}(T)$ (i.e., $y = T(x)$), hence $\text{graph}(T)$ is a closed set. See Thm. 22 below for a more formal statement.

Theorem 19 (Closed Graph Theorem, Royden version/proof)**.** *Let $T : X \to Y$ be a linear operator and X, Y Banach. Then T is continuous (i.e., bounded) if and only if it is closed.*

Proof. If T is continuous, it is sequentially continuous, and hence closed. Now we suppose the converse, that T is closed. Introduce a new norm $\|\cdot\|_*$ on X by

$$\|x\|_* = \|x\| + \|T(x)\| \quad \forall x \in X.$$

Observe that the closedness of the operator T is equivalent to the completeness of the normed linear space $(X, \|\cdot\|_*)$, therefore we have by assumption that $(X, \|\cdot\|_*)$ is Banach. [3]

Now, note that $\|\cdot\| \leq \|\cdot\|_*$ by definition. By the corollary above, that means those norms are equivalent, so there is some constant c such that $\|x\|_* \leq c\|x\|$ for all x. But this implies that $\|T(x)\| \leq c\|x\|$, hence T is bounded, hence it is continuous since it is linear. □

5.1.2 Other versions (with more details filled in)

Let X and Y be normed linear spaces. Then $X \times Y$ is a normed linear space with the two operations defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad \alpha\,(x, y) = (\alpha\, x, \alpha\, y),$$

and the norm defined by

$$\|(x, y)\| = \|x\| + \|y\|.$$

Lemma 20. *If X and Y are Banach spaces, then $X \times Y$ is also a Banach space.*

Proof. Let $\{z_n\}$ be a Cauchy sequence in $X \times Y$. Here $z_n = (x_n, y_n)$. Let $\epsilon > 0$ be given. Then there is N such that

$$\|x_n - x_m\| + \|y_n - y_m\| = \|z_n - z_m\| < \epsilon, \quad \forall\, m, n > N.$$

So, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in X and Y respectively. Since X and Y are complete, there are $x \in X$ and $y \in Y$ such that $x_n \to x$ and $y_n \to y$. Define $z = (x, y) \in X \times Y$. Letting $m \to \infty$ in the above inequality,

$$\|x_n - x\| + \|y_n - y\| < \epsilon, \quad \forall\, n > N.$$

This implies $\|z_n - z\| < \epsilon$ for all $n > N$. Hence $z_n \to z$. □

Now we give the definition of closed linear operator.

[3] We prove one side of this statement rigorously. Let $(x_n) \subset X$ be Cauchy with this new norm, which also implies it is Cauchy with the original norm, and since X is complete in this original norm, there is some $x \in X$ with $x_n \to x$. Similarly, the sequence (Tx_n) is Cauchy in Y with the original norm in Y, and since Y is Banach there is some $y \in Y$ with $T(x_n) \to y$. By the assumption that T is closed, $y = T(x)$, this implies that $x_n \to x$ in the $\|\cdot\|_*$ norm since $\|x_n - x\|_* = \|x_n - x\| + \|T(x_n - x)\| = \|x_n - x\| + \|T(x_n) - y\|$.

Definition 21. *Let X and Y be normed linear spaces and let $D \subset X$ be a subspace. A linear operator $T : D \to Y$ is called* ***closed*** *if its graph*

$$\text{graph}(T) = \{(x, y) \mid x \in D,\ y = Tx\}$$

is closed in $X \times Y$.

Theorem 22. *Let X and Y be normed linear spaces and let $D \subset X$ be a subspace. Then a linear operator $T : D \to Y$ is* ***closed*** *if and only if it has the following property: If $x_n \to x$ and $Tx_n \to y$, then $x \in D$ and $y = Tx$.*

Proof. By definition, graph(T) is closed if and only if $\overline{\text{graph}(T)} \subset \text{graph}(T)$. Assume T is closed. If $x_n \to x$ and $Tx_n \to y$, then $(x, y) \in \overline{\text{graph}(T)}$. Thus, $(x, y) \in \text{graph}(T)$ since graph(T) is closed. That is $x \in D$ and $y = Tx$. Conversely, assume that if $x_n \to x$ and $Tx_n \to y$, then $x \in D$ and $y = Tx$. Let $z = (x, y) \in \overline{\text{graph}(T)}$. Then there is $z_n = (x_n, Tx_n)$ such that $z_n \to z$. Thus, we have $x_n \to x$ and $Tx_n \to y$. By assumption, $x \in D$ and $y = Tx$. Hence $z \in \text{graph}(T)$. □

Now we have the following important closed graph theorem, which gives a condition for when a closed linear operator is bounded.

Theorem 23 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and let $D \subset X$ be a subspace. Let $T : D \to Y$ be a closed linear operator. If D is closed in X, then T is bounded.*

Proof. Since graph(T) is closed in $X \times Y$ and D is closed in X, then graph(T) and D are complete since they are inside complete spaces. Consider the mapping $P : \text{graph}(T) \to D$ defined by

$$P(x, Tx) = x.$$

Clearly, P is linear. Also P is bounded since

$$\|P(x, Tx)\| = \|x\| \le \|x\| + \|Tx\| = \|(x, Tx)\|.$$

So, P is a bounded linear operator. Moreover, it is clear that P is bijective with the inverse defined by

$$P^{-1}(x) = (x, Tx).$$

Then, by the open mapping theorem (16), the inverse operator P^{-1} is bounded. So, there is a real number b such that $\|P^{-1}x\| \le b\|x\|$. Thus,

$$\|Tx\| \le \|Tx\| + \|x\| = \|(x, Tx)\| = \|P^{-1}x\| \le b\|x\|.$$

Hence T is bounded. □

6 Axiom of choice and Zorn's lemma

The proof of the Hahn-Banach and Baire Category theorem rely on Zorn's lemma, which is an equivalent version of the axiom of choice "AC" (note that Hahn-Banach and Baire Category do not imply the AC, so they are not equivalent to it).

The axiom of choice seems harmless at first. It is an axiom, meaning that we *assume* it is true (and this is referred to as ZFC, for Zermelo-Frankel set theory with the axiom of Choice; ZF does not assume the AC). It says that if we have a collection of sets, $X = \{Y_\alpha \mid \alpha \in \mathcal{A}\}$ for some arbitrary (e.g., uncountable) index set $\mathcal{A}$, where each Y_α is itself a set, then there exists some "choice" function that can select a single element from each set Y_α. That is, there is some f such that $f(Y_\alpha) \in Y_\alpha$.

For most sets, this is dead-obvious. E.g., if each Y_α was a collection of integers, we could just select the smallest integer. Or if it was an interval on the real line, we could select the midpoint of the interval. But in general, we cannot specify what is selected, we just assume that it can be done.

This has extremely bizarre implications. It implies that the real numbers are well-ordered. We say that a non-empty set is well-ordered if every subset of it has a least element. For example, the natural numbers are well-ordered. With the real numbers, using the usual ordering (where $1 < 2 < 2.034$ etc.,), this is not true, since an open interval like $(0, 1)$ has no least element (that is, there is an inf but not a min). Under the AC, it means we believe that there is some other ordering of the real numbers such that there always is a least element.

It also implies the Banach-Tarksi paradox. Take a 3D solid ball, decompose it into a finite number of disjoint subsets, then these can be put back (that is, moved and rotated) to form 2 identical copies of the ball. This is done by making very weird subsets (think of things like several copies of the Cantor set). And of course it is not true in our usual physical world, hence we call it a paradox.

Despite these oddities, the AC is necessary for most powerful theorems, so we implicitly use it all the time!