Section 3.3: Fredholm Integral Equations

Suppose that $k : [a, b] \times [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are given functions and that we wish to find an $f : [a, b] \to \mathbb{R}$ that satisfies

$$f(x) = g(x) + \int_a^b k(x, y)\, f(y)\, dy. \tag{1}$$

Equation (1) is known as a **Fredholm Integral Equation** (F.I.E.) or a Fredholm Integral Equation "of the second kind". (F.I.E.'s of the "first kind" have $g(x) = 0$.) The function k is referred to as the "integral kernel".

The F.I.E. may be written as a fixed point equation

$$Tf = f$$

where the operator T is defined by

$$Tf(x) = g(x) + \int_a^b k(x, y)\, f(y)\, dy.$$

Theorem: If $k : [a, b] \times [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and if

$$\sup_{a \le x \le b} \int_a^b |k(x, y)|\, dy < 1,$$

there exists a unique continuous $f : [a, b] \to \mathbb{R}$ that satisfies the Fredholm integral equation.

Proof:

- We will show that the sup condition implies that T is a contraction mapping in $\mathcal{C}([a, b])$ (equipped with the usual uniform/sup norm). Then, since $\mathcal{C}([a, b])$ is complete, we can use the Contraction Mapping Theorem to show that there exists a unique fixed point $f \in \mathcal{C}([a, b])$.
- Note that

$$\begin{aligned} ||Tf_1 - Tf_2|| &= \sup_{a \le x \le b} |Tf_1(x) - Tf_2(x)| \\ &= \sup_{a \le x \le b} \left| \int_a^b k(x, y)(f_1(y) - f_2(y))\, dy \right| \\ &\le \sup_{a \le x \le b} \int_a^b |k(x, y)| \cdot |(f_1(y) - f_2(y))|\, dy \\ &\le ||f_1 - f_2|| \underbrace{\sup_{a \le x \le b} \int_a^b |k(x, y)|\, dy}_{<1} \end{aligned}$$

  So, T is a contraction mapping.

- By the Contraction Mapping Theorem, the equation $Tf = f$, and therefore the F.I.E., has a unique solution in $\mathcal{C}([a,b])$. $\square$

We now know that, if the conditions of the previous theorem are satisfied, we may solve (**??**) by choosing any $f_0 = \mathcal{C}([a,b])$ and computing

$$f = \lim_{n\to\infty} T^n f_0.$$

The **Fredholm Integral Operator**, denoted by K, is defined as on functions $f \in \mathcal{C}([a,b])$ as

$$Kf := \int_a^b \not{k}(x,y)\, f(y)\, dy$$

where $\not{k}$ is an F.I.E. kernel. Note that K is a linear operator.

The F.I.E. is then written

$$f = g + Kf$$

which can also be written

$$Tf = g + Kf$$

using the fixed point equation $Tf = f$.

Note that

$$\begin{aligned}
Tf_0 &= g + Kf_0 \\
T^2 f_0 &= T(Tf_0) = T(g + Kf_0) = g + K(g + Kf_0) = g + Kg + K^2 f_0 \\
T^3 f_0 &= T(T^2 f_0) = g + Kg + K^2 g + K^3 f_0 \\
&\vdots \\
T^n f_0 &= g + Kg + K^2 g^2 + \cdots + K^{n-1} g^{n-1} + K^n f_0
\end{aligned}$$

On HW 6, we will see that $\lim_{n\to\infty} K^n f_0 = 0$.

Thus

$$f = \lim_{n\to\infty} T^n f_0 = \sum_{n=0}^{\infty} K^n g.$$

Example: Solve the Fredholm Integral Equation

$$f(x) = 1 + \int_0^1 x\, f(y)\, dy.$$

Note that

$$\sup_{a \le x \le b} \int_a^b |k(x,y)|\, dy = \sup_{0 \le x \le 1} \int_0^1 x\, dy = 1.$$

We need this strictly less than 1 in order to use our Theorem from page 1. To this end, we will "back off of 1" a little bit and consider solving.

$$f(x) = 1 + \int_0^\alpha x\, f(y)\, dy \tag{2}$$

for $0 < \alpha < 1$.

Note that now

$$\sup_{a \le x \le b} \int_a^b |k(x,y)|\, dy = \sup_{0 \le x \le \alpha} \int_0^\alpha x\, dy = \alpha^2 < 1.$$

Furthermore $g(x) = 1$ and $k(x,y) = x$ are continuous functions on $[0,1]$ so all of the conditions of the Theorem on page 1 are satisfied.

So, we may start with any $f_0 \in \mathcal{C}([0,1])$ and repeatedly apply T where $Tf(x) = 1 + \int_0^\alpha x\, f(y)\, dy$.

Let $f_0(x) = 1$. Then

$$f_1(x) = Tf_0(x) = 1 + \int_0^\alpha x\, f_0(y)\, dy = 1 + \int_0^\alpha x\, dy = 1 + \alpha x,$$

and

$$\begin{aligned} f_2(x) &= Tf_1(x) = 1 + \textstyle\int_0^\alpha x\, f_1(y)\, dy = 1 + \int_0^\alpha x(1+\alpha y)\, dy \\ &= 1 + x\left[\alpha + \tfrac{1}{2}\alpha^3\right] \end{aligned}$$

and

$$\begin{aligned} f_3(x) &= Tf_2(x) = 1 + \textstyle\int_0^\alpha x\, f_2(y)\, dy = 1 + \int_0^\alpha x[1 + y(\alpha + \frac{1}{2}\alpha^3)]\, dy \\ &= 1 + x\left[\alpha + \tfrac{1}{2}\alpha^3 + \tfrac{1}{2^2}\alpha^5\right]. \end{aligned}$$

Continuing, we get

$$f_n(x) = 1 + x\left[\alpha + \frac{1}{2}\alpha^3 + \frac{1}{2^2}\alpha^5 + \cdots + \frac{1}{2^{n-1}}\alpha^{2^{n-1}+1}\right].$$

Therefore,

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) = 1 + x \textstyle\sum_{n=0}^\infty \frac{1}{2^n}\alpha^{2n+1} \\ &= 1 + x\alpha \textstyle\sum_{n=0}^\infty \left(\frac{\alpha^2}{2}\right)^n = 1 + x\alpha \cdot \frac{1}{1-\alpha^2/2} \\ &= 1 + \tfrac{2\alpha}{2-\alpha^2}\, x \end{aligned}$$

It is easy to check that this satisfies the given F.I.E. Note that the sum in that second to last line is still convergent for $\alpha \in (-\sqrt{2}, \sqrt{2})$ and furthermore that the solution satisfies (2) for any $\alpha \neq \pm\sqrt{2}$!

Example: Solve the Fredholm Integral Equation

$$f(x) = \sin x + \int_0^{\pi/2} \sin x \cos y \, f(y) \, dy.$$

Note first that

$$\sup_{a \le x \le b} \int_a^b |\not{k}(x, y)| \, dy = \sup_{0 \le x \le \pi/2} \int_0^{\pi/2} |\sin x \cos y| \, dy$$

$$= \sup_{0 \le x \le \pi/2} \int_0^{\pi/2} \sin x \cos y \, dy = \sup_{0 \le x \le \pi/2} \sin x = \sin(\pi/2) = 1 \not< 1$$

However, in light of the comments at the end of the previous example, we are going to try to leave the $\pi/2$ in place.

Let $f_0(x) = 1$.

Then

$$\begin{aligned} f_1(x) &= \sin x + \textstyle\int_0^{\pi/2} \sin x \cos y \cdot f_0(y) \, dy \\ &= \sin x + \textstyle\int_0^{\pi/2} \sin x \cos y \cdot 1 \, dy \\ &= \sin x + \sin x \textstyle\int_0^{\pi/2} \cos y \, dy \\ &= \sin x + \sin x \cdot 1 = 2 \sin x. \end{aligned}$$

Now,

$$\begin{aligned} f_2(x) &= \sin x + \textstyle\int_0^{\pi/2} \sin x \cos y \cdot f_1(y) \, dy \\ &= \sin x + \textstyle\int_0^{\pi/2} \sin x \cos y \cdot 2 \sin y \, dy \\ &= \sin x + 2 \sin x \textstyle\int_0^{\pi/2} \underbrace{\sin y}_{u} \underbrace{\cos y \, dy}_{du} \\ &= \sin x + 2 \sin x \cdot \tfrac{1}{2} \sin^2 y \Big|_0^{\pi/2} \\ &= \sin x + 2 \sin x \cdot \tfrac{1}{2} = 2 \sin x. \end{aligned}$$

So, we have already reached our fixed point! That is, $f_n(x) = 2 \sin x$ for $n = 1, 2, \ldots$. Thus, we have

$$f(x) = \lim_{n \to \infty} f_n(x) = 2 \sin x.$$

It is easy to see/verify that this satisfies the given F.I.E.

Section 3.4: Boundary Value Problems

In this section we wish to find solutions to the **boundary value problem** (BVP) given by

$$\boxed{\begin{array}{l} u''(x) = q(x)u(x), \quad 0 < x < 1 \\ \\ u(0) = u_0, \;\; u(1) = u_1 \end{array}}$$

When $q(x)$ is constant, the solution is easy. Recall that for a second order differential equation of the form

$$au''(x) + bu'(x) + cu(x) = 0$$

one first finds roots r_1 and r_2 for the **auxiliary equation**

$$au^2 + bu + c = 0.$$

Then

- If r_1 and r_2 are **real and distinct**, the solution has the form

$$u(x) = c_1 e^{r_1 x} + c2e^{r_2 x}.$$

- If the roots are **real and repeated** ($r_1 = r_2 = r$), the solution has the form

$$u(x) = c_1 e^{rx} + c_2 x e^{rx}.$$

- If the roots are **complex** ($r_1 = a + ib$, $r_2 = a - ib$), the solution has the form

$$u(x) = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx).$$

Non-constant $q(x)$ is more difficult and is the point of this Section.

To solve

$$u''(x) = q(x)u(x), \quad 0 < x < 1$$

$$u(0) = u_0, \;\; u(1) = u_1$$

we begin by zeroing out the boundary conditions and considering the function

$$v(x) := u(x) - u_0 + (u_0 - u_1)x.$$

Note that $v''(x) = u''(x)$ and that

$$q(x)u(x) = q(x)v(x) + q(x)[u_0 + (u_1 - u_0)x].$$

Our new BVP is given by

$$\boxed{\begin{array}{l} v''(x) = q(x)v(x) + q(x)[u_0 + (u_1 - u_0)x] \\[1em] v(0) = 0, \;\; v(1) = 0 \end{array}}$$

On our road to a solution, we first consider something simpler.

Theorem: Let $f : [0,1] \to \mathbb{R}$ be continuous.
The unique solution of the BVP

$$v''(x) = -f(x), \qquad v(0) = 0, \; v(1) = 0$$

is given by

$$v(x) = \int_0^1 g(x,y)\, f(y)\, dy$$

where

$$g(x,y) = \begin{cases} x(1-y) & , \quad 0 \le x \le y \le 1 \\ y(1-x) & , \quad 0 \le y \le x \le 1. \end{cases}$$

Proof:

- Note that

$$v''(s) = -f(s) \qquad \Rightarrow \qquad \int_1^y v''(s)\, ds = -\int_1^y f(s)\, ds$$

$$\Rightarrow \quad v'(y) = -\int_1^y f(s)\, ds + c_1$$

- So,

$$\int_0^x v'(y)\, dy = -\int_0^x \int_1^y f(s)\, ds\, dy + c_1 x$$

which $\Rightarrow v(x) = -\int_0^x \int_1^y f(s)\, ds\, dy + c_1 x + c_2$.

- Integrating by parts with "u" $= \int_1^y f(s)\,ds$ and "dv" $= dy$ (so "du" $= f(y)\,dy$ and "v" $= y$) we get

$$\begin{aligned} v(x) &= -\left\{[y \textstyle\int_1^y f(s)\,ds]_{y=0}^{y=x} - \int_0^x y f(y)\,dy\right\} + c_1 x + c_2 \\ &= -\left[x \textstyle\int_1^x f(s)\,ds - \int_0^x y f(y)\,dy\right] + c_1 x + c_2 \\ &= -\left[x \textstyle\int_1^x f(y)\,dy - \int_0^x y f(y)\,dy\right] + c_1 x + c_2 \\ &= -\left[-x \textstyle\int_x^1 f(y)\,dy - \int_0^x y f(y)\,dy\right] + c_1 x + c_2 \end{aligned}$$

  for $x \in [0,1]$.

- Now the boundary conditions give

$$\begin{aligned} v(0) &= c_2 = 0 \\ v(1) &= \textstyle\int_0^1 y f(y)\,dy + c_1 = 0 \quad \Rightarrow \quad c_1 = -\int_0^1 y f(y)\,dy \end{aligned}$$

- Thus, we have that

$$\begin{aligned} v(x) &= x \textstyle\int_x^1 f(y)\,dy + \int_0^x y f(y)\,dy - \int_0^1 y f(y)\,dy \\ &= \textstyle\int_0^x y(1-x) f(y)\,dy + \int_x^1 x(1-y) f(y)\,dy \\ &= \textstyle\int_0^1 g(x,y) f(y)\,dy \quad \checkmark \end{aligned}$$