Supplement/Review on Fourier Series
APPM 5450 Spring 2018 Applied Analysis 2

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1. Define the torus $\mathbb{T} = [0, 2\pi]/(2\pi\mathbb{Z})$, which is a compact set. Note $C(\mathbb{T}) \subsetneq C([0, 2\pi])$.

2. For a *compact* set $\Omega \subset \mathbb{R}^n$, $L^p(\Omega) \subset L^q(\Omega)$ if $1 \le q \le p \le \infty$, e.g., $L^\infty(\Omega) \subset L^2(\Omega) \subset L^1(\Omega)$. In particular, this holds if we take $n = 1$ and $\Omega = \mathbb{T}$. These inclusions are not true if Ω is not compact. [Question to ponder: if L^∞ is not separable and L^1 is, how is it possible for $L^\infty \subset L^1$?]

3. The Fourier Basis $(e_n)_{n \in Z}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{T})$. Note we take $n \in \mathbb{Z}$ not just $n \in \mathbb{N}$. Define $e_n = 1/\sqrt{2\pi} \cdot e^{inx}$.

4. For a function f defined on the torus, the nth Fourier coefficient is

$$\widehat{f}_n = \langle e_n, f \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx}\, dx$$

   Hence, if $f \in L^1(\mathbb{T})$, then $\widehat{f}_n$ is finite. We often require the stricter condition $f \in L^2(\mathbb{T})$ since L^2 is a Hilbert space. The Hunter and Nachtergaele book mainly works in $L^2(\mathbb{T})$.

5. When we speak of $L^2(\mathbb{T})$, we generally mean functions f that map $\mathbb{T}$ to $\mathbb{C}$, rather than mapping $\mathbb{T}$ to $\mathbb{R}$. The field in our Hilbert space L^2 is likewise $\mathbb{C}$ and not $\mathbb{R}$. If we wanted to work with the real field, we'd need to switch to sine/cosine bases, since e^{-inx} is obviously not real-valued.

1 Density

Density arguments are very useful. Here are some examples of how we use them along with some useful facts.

1. If K is a compact subset of R^n, then $(C(K), \|\cdot\|_\infty)$ is a Banach space (Thm. 2.4). Similarly, the set of bounded continuous functions on all of $\mathbb{R}^n$, $C_b(R^n)$, is also Banach with the uniform norm.

2. The set of polynomials $\mathbb{P}([a, b])$ is dense in $C([a, b])$ with respect to the uniform norm (Thm. 2.9 Weierstrass).

3. $\mathbb{P}([a, b])$ is a subspace of $C([a, b])$ but it is not closed (since its closure is $C([a, b])$).

4. Defining periodic functions in L^2 is tricky, since a periodic function requires point values, something that functions in L^2 do not have (since they are only defined up to a set of measure zero). Therefore we define $L^2(\mathbb{T})$ as the completion of $C(\mathbb{T})$ with the L^2 norm. Hence, by definition, $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.

5. Let $\mathcal{P} = \mathrm{span}(e_n)$ be the set of trigonometric polynomials. To prove (e_n) is an orthonormal basis for $L^2(\mathbb{T})$, one must show $\mathcal{P}$ is dense in L^2 so that (e_n) is total. This fact is established in §7.1.

6. $\mathcal{P}$ is dense in $C(\mathbb{T})$ with respect to the uniform norm (Thm. 7.3), and hence also with respect to the L^2 norm.

7. The space of continuous functions with compact support, $C_c(\mathbb{R}^n)$, is dense in $L^p(\mathbb{R}^n)$, with respect to the L^p norm, for $1 \le p < \infty$; Theorem 12.50. This is not true for L^∞ since that space is not separable.
8. In fact, not only is C_c dense in L^p, but C_c^∞ is dense in L^p as well ($p = \infty$ excluded still); Theorem 12.51.

2 Convergence of Fourier Series

Let $f \in L^1(T)$ and define the partial sums

$$S_N = \sum_{n=-N}^{N} \widehat{f}_n e_n.$$

We can ask, when does $\lim_{N\to\infty} S_N = f$, and in what sense? These results are collected from a few places, including our book and Grafakos "Classical Fourier Analysis" (Springer).

1. $\lim_{N\to\infty} \|S_N - f\|_{L^2} = 0$ if and only if $f \in L^2(\mathbb{T})$. This is the Riesz-Fisher Theorem from 1907. In our modern theory, we would say that this follows trivially since we have proven that (e_n) is an orthonormal basis (see o.n.b. handout).
2. More generally, for $1 < p < \infty$, $\lim_{N\to\infty} \|S_N - f\|_{L^p} = 0$ for $f \in L^p(\mathbb{T})$.
3. It is **not true** that $\lim_{N\to\infty} \|S_N - f\|_{L^1} = 0$ for all $f \in L^1(\mathbb{T})$. A counter-example was provided by Kolmogorov in 1923. It is also not true for L^∞.
4. Do we have **pointwise convergence**?
   a) For $f \in L^p(\mathbb{T})$ this doesn't even make sense because these functions are not defined pointwise.
   b) For $f \in C(\mathbb{T})$, the question does make sense. The answer is **no** (Paul du Bois-Raymond, 1876), but it is not easy to find such continuous functions. However, in some sense these functions are "typical" since for any fixed x, the set of continuous functions whose Fourier series diverges at x is dense in $C(\mathbb{T})$.
   c) For $C^1(\mathbb{T})$, the answer is yes (but in this case we can even say the convergence is uniform).
   d) More generally, if a function is differentiable at x, then there is pointwise convergence at this x
   e) There are weaker conditions than differentiability that will guarantee convergence, but we do not discuss them here
5. Do we have **pointwise convergence almost everywhere**?
   a) If $f \in L^2(\mathbb{T})$, **yes**! This was Luzan/Lusin's conjecture, proved true by Carleson in 1966, so it's fair to say it is not obvious. It's actually true for $f \in L^p(\mathbb{T})$ for all $p > 1$ but not for $p = 1$.
   b) Since $C(\mathbb{T}) \subset L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$, this is also true for continuous functions
   c) It is not true in $L^1(\mathbb{T})$, and Kolmogorov's counter-example mentioned above can be used here as well.
6. If we have pointwise convergence, is it **uniform**? Yes, if $f \in C^1(\mathbb{T})$, but no if just $f \in C(\mathbb{T})$ since in $C(\mathbb{T})$ there are functions that are not even the pointwise limit of their partial sums. Even if you take $f \in C(\mathbb{T})$ *and* assume there is pointwise convergence, it need not necessarily be uniform.
   a) Again, there are weaker conditions than C^1 that will guarantee uniform convergence, but we do not discuss them here.
7. If we assume f is piece-wise in $C^1(\mathbb{T})$ (that is, we can divide $\mathbb{T}$ into a finite number of intervals, and on each such interval, f is C^1), then inside the smooth regions one might expect uniform convergence. Is this true? That is, even if we exclude the points of discontinuity, do we have uniform convergence? (i.e, uniform convergence almost everywhere). The answer is No, we do *not* get uniform convergence. This is the **Gibbs phenomenon**.