APPM 4/5560 Markov Processes

Fall 2019, Final Exam Review Problems Solutions

1. We want to find $\pi_0 + \pi_2$.

$$\begin{aligned} \lambda_i &= \lambda, \qquad i = 0, 1, 2, \ldots \\ \mu_1 &= \mu \\ \mu_i &= 2\mu, \qquad i = 2, 3, \ldots \end{aligned}$$

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \pi_0 = \left(\frac{\lambda}{\mu}\right)^n \left(\frac{1}{2}\right)^{n-1} \pi_0$$

which holds for $n = 1, 2, \ldots$.

To find π_0:

$$1 = \sum_{n=0}^{\infty} \pi_0 \left[1 + \frac{\lambda}{\mu} \sum_{n=1}^{\infty} \left(\frac{\lambda}{2\mu}\right)^{n-1}\right] = \pi_0 \left[1 + \frac{\lambda/\mu}{1 - \lambda/(2\mu)}\right] = \pi_0 \frac{2\mu + \lambda}{2\mu - \lambda}$$

assuming that $\lambda/(2\mu) < \infty$.

So, we have

$$\pi_0 = \frac{2\mu - \lambda}{2\mu + \lambda}.$$

We then have

$$\pi_1 = \frac{\lambda}{\mu} \pi_0 = \frac{\lambda}{\mu} \frac{2\mu - \lambda}{2\mu + \lambda}.$$

The probability that at most one server is busy is

$$\pi_0 + \pi_1 = \frac{2\mu - \lambda}{2\mu + \lambda} \left[1 + \frac{\lambda}{\mu}\right]$$

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2. We want

$$0.95 = \sum_{n=0}^{9} \pi_n$$

where

$$\pi_n = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n, \qquad n = 0, 1, 2, \ldots$$

and $\lambda = 3$.

$$\sum_{n=0}^{9} \pi_n = \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{9} \left(\frac{\lambda}{\mu}\right)^n = 1 - \left(\frac{\lambda}{\mu}\right)^{10}$$

So, we need to solve

$$1 - \left(\frac{3}{\mu}\right)^{10} = 0.95$$

for μ. The answer is $\mu = 3/(0.05)^{1/10} \approx 4.048$.

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3. $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = 0$, $\lambda_4 = 0, \ldots$, etc...

   $\mu_1 = \mu_2 = \mu_3 = \lambda$

   So the stationary distribution which gives the probability of there being n customers in the shop is, for $n = 0, 1, 2, 3$

$$\pi_n = \frac{\lambda_0 \cdots \lambda_n}{\mu_1 \cdots \mu_n} \pi_0 = \left(\frac{\lambda}{\lambda}\right)^n \pi_0 = \pi_0$$

   So,

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 = \pi_0 + \pi_0 + \pi_0 + \pi_0 = 4\pi_0$$

$$\Rightarrow \pi_0 = \frac{1}{4}$$

   and hence $\pi_0 = \pi_1 = \pi_2 = \pi_3 = 1/4$.

   So, the expected number of customers in the shop is

$$\sum_{n=0}^{3} n \cdot P(n \text{ customers in the shop}) = \sum_{n=0}^{3} n\pi_n = \frac{1}{4}\sum_{n=0}^{3} n = \frac{1}{4}[0 + 1 + 2 + 3]$$

$$= \frac{3}{2}$$

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4. I'm going to interpret the question as the amount of time from the start of one idle period to the start of the next idle period. In this case, we want to find

$$\mathsf{E}[I] + \mathsf{E}[B]$$

   where I is a typical idle period and B is a typical busy period for the server.

   We know that $\mathsf{E}[I] = 1/\lambda$.

   As for the busy period, let S be the service time for the customer who starts the busy period. We have

$$\begin{aligned} \mathsf{E}[B] &= \textstyle\int_0^\infty \mathsf{E}[B|S = s] \cdot f(s)\, ds \\ &= \textstyle\int_0^\infty \mathsf{E}[B|S = s] \cdot \mu e^{-\mu s}\, ds \end{aligned}$$

The first customer, with service time $S = s$ starts a busy period for the server that is at least s units of time long. If more customers come in during this service time, the busy period will increase.

Let N be the number of arrivals during this first customer's service period. To find $\mathsf{E}[B|S = s]$ we will condition on N. We have

$$\mathsf{E}[B|S = s] = \sum_{n=0}^{\infty} \mathsf{E}[B|S = s, N = n] \cdot P(N = n|S = s).$$

Due to the Poisson arrival process, $P(N = n|S = s) = \frac{e^{-\lambda s}(\lambda s)^n}{n!}$. Let's examine the other part. We have

$$\mathsf{E}[B|S = s, N = 0] = s$$

since if no other customers arrive while the first customer is served, the server's busy period ends when the first customer leaves and the busy period is just that first customer's service time.

Now, if exactly 1 customer arrives during the first customer's service time, the busy period of the server will be the original s units of time on the first customer plus another busy period that will consist of the second customer's service time plus the service times of whoever else came in during the second customer's service time. All of this just becomes a new busy period started by the second customer. We have

$$\mathsf{E}[B|S = s, N = 1] = s + \mathsf{E}[B]$$

Similarly, we have

$$\begin{aligned} \mathsf{E}[B|S = s, N = 2] &= s + 2\mathsf{E}[B] \\ &\vdots \\ \mathsf{E}[B|S = s, N = n] &= s + n\mathsf{E}[B] \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathsf{E}[B|S = s] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[B|S = s, N = n] \cdot P(N = n|S = s) \\ &= \textstyle\sum_{n=0}^{\infty} [s + n\mathsf{E}[B]] \frac{e^{-\lambda s}(\lambda s)^n}{n!} \\ &= s \underbrace{\sum_{n=0}^{\infty} \frac{e^{-\lambda s}(\lambda s)^n}{n!}}_{1} + \mathsf{E}[B] \underbrace{\sum_{n=0}^{\infty} n \frac{e^{-\lambda s}(\lambda s)^n}{n!}}_{\mathsf{E}[Poisson(\lambda s)]} \\ &= s + \lambda s \mathsf{E}[B] \end{aligned}$$

Finally, going back to the beginning of this solution, we have

$$\begin{aligned}
\mathsf{E}[B] &= \textstyle\int_0^\infty \mathsf{E}[B|S=s]\cdot \mu e^{-\mu s}\,ds \\
&= \textstyle\int_0^\infty [s+\lambda s\mathsf{E}[B]]\cdot \mu e^{-\mu s}\,ds \\
&= [1+\lambda\mathsf{E}[B]]\underbrace{\int_0^\infty s\,\mu e^{-\mu s}\,ds}_{\mathsf{E}[exp(rate=\mu)]} \\
&= [1+\lambda\mathsf{E}[B]]\cdot \tfrac{1}{\mu}
\end{aligned}$$

We have

$$\mathsf{E}[B] = [1+\lambda\mathsf{E}[B]]\cdot\frac{1}{\mu}.$$

Solving for $\mathsf{E}[B]$ gives

$$\mathsf{E}[B] = \frac{1}{\mu-\lambda}$$

The final answer is

$$\boxed{\mathsf{E}[I]+\mathsf{E}[B] = \frac{1}{\lambda}+\frac{1}{\mu-\lambda} = \frac{\mu}{\lambda(\mu-\lambda)}.}$$

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5. No, $\{N(t)\}$ does not have a stationary distribution. It is an increasing process, so there is no way that the distribution of its values at time time t_1 could be the same as the distribution of its values at some later time t_2. To express this a bit more formally, one could write down the generator matrix Q and then it is easy to see that, for $\lambda > 0$ the only solution to $\vec{\pi}Q = \vec{0}$ is $\vec{\pi} = \vec{0}$ which is not a distrbution. (It doesn't sum to 1.)

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6. (a) Birth rates: $\lambda_0 = d\lambda$, $\lambda_1 = (d-1)\lambda$, etc... In general,

$$\lambda_i = \begin{cases} (d-i)\lambda & , \quad i = 0,1,\ldots,d \\ 0 & , \quad \text{otherwise} \end{cases}$$

Death Rates: $\mu_0 = 1$, $\mu_1 = \mu$, $\mu_2 = 2\mu$, etc... In general,

$$\mu_i = \begin{cases} i\mu & , \quad i = 0,1,\ldots,d \\ 0 & , \quad \text{otherwise} \end{cases}$$

(b) $\pi_n = 0$ for $n > d$. For $n = 0, 1, 2, \ldots, d$,

$$\begin{aligned}
\pi_n &= \tfrac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\,\pi_0 \\
&= \tfrac{(d\lambda)((d-1)\lambda)\cdots((d-(n-1))\lambda)}{(\mu)(2\mu)\cdots(n\mu)}\,\pi_0 \\
&= \left(\tfrac{\lambda}{\mu}\right)^n \tfrac{d!/(d-n)!}{n!}\,\pi_0 \\
&= \left(\tfrac{\lambda}{\mu}\right)^n \tfrac{d!}{n!(d-n)!}\,\pi_0 \\
&= \binom{d}{n}\left(\tfrac{\lambda}{\mu}\right)^n \pi_0
\end{aligned}$$

To find π_0:

$$1 = \pi_0 \sum_{n=0}^{d} \binom{d}{n}\left(\frac{\lambda}{\mu}\right)^n = \pi_0 \sum_{n=0}^{d} \binom{d}{n}\left(\frac{\lambda}{\mu}\right)^n (1)^{d-n} = \pi_0 \left(1 + \frac{\lambda}{\mu}\right)^d$$

where the last equality is by the binomial theorem. Therefore

$$\pi_0 = \left[\left(1 + \frac{\lambda}{\mu}\right)^d\right]^{-1}$$

and, for $n = 1, 2, \ldots, d$,

$$\pi_n = \binom{d}{n}\left(\frac{\lambda}{\mu}\right)^n \pi_0.$$

(c)

$$\begin{aligned}
\mathsf{E}[X(t)] &= \textstyle\sum_{n=0}^{d} n\,\pi_n \\
&= \pi_0 \textstyle\sum_{n=0}^{d} n \binom{d}{n}\left(\tfrac{\lambda}{\mu}\right)^n \\
&= \pi_0 \textstyle\sum_{n=1}^{d} n \binom{d}{n}\left(\tfrac{\lambda}{\mu}\right)^n \\
&= \pi_0\,(\lambda/\mu) \textstyle\sum_{n=1}^{d} n \binom{d}{n}\left(\tfrac{\lambda}{\mu}\right)^{n-1} \\
&= \pi_0\,(\lambda/\mu)\tfrac{d}{dr} \textstyle\sum_{n=1}^{d} \binom{d}{n} r^n
\end{aligned}$$

where $r = \lambda/\mu$.

Now

$$\sum_{n=1}^{d} \left(\begin{array}{c} d \\ n \end{array} \right) r^n = \sum_{n=0}^{d} \left(\begin{array}{c} d \\ n \end{array} \right) r^n - 1 = (r+1)^d - 1.$$

So,

$$\frac{d}{dr} \sum_{n=1}^{d} \left(\begin{array}{c} d \\ n \end{array} \right) r^n = \frac{d}{dr} \left[(r+1)^d - 1 \right] = d(r+1)^{d-1}.$$

Therefore

$$\begin{aligned} \mathsf{E}[X(t)] &= \pi_0(\lambda/\mu)\, d \left(\tfrac{\lambda}{\mu} + 1\right)^{d-1} \\ &= \left[\left(1 + \tfrac{\lambda}{\mu}\right)^d\right]^{-1} (\lambda/\mu)\, d \left(\tfrac{\lambda}{\mu} + 1\right)^{d-1} \\ &= \tfrac{d \cdot \lambda/\mu}{1+\lambda/\mu} \end{aligned}$$

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7. (a) Let $X =$ the number of people in the queue in equilibrium. Then

$$P(X = n) = \pi_n = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n, \qquad n = 0, 1, 2, \ldots$$

This is a geometric distribution so you could just quote the mean of the geometric. Alternatively,

$$\begin{aligned} \mathsf{E}[X] &= \textstyle\sum_{n=0}^{\infty} n \cdot P(X = n) \\ &= \left(1 - \tfrac{\lambda}{\mu}\right) \textstyle\sum_{n=0}^{\infty} n \left(\tfrac{\lambda}{\mu}\right)^n \\ &= \left(1 - \tfrac{\lambda}{\mu}\right) \textstyle\sum_{n=1}^{\infty} n \left(\tfrac{\lambda}{\mu}\right)^n \\ &= \left(1 - \tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \textstyle\sum_{n=1}^{\infty} n \left(\tfrac{\lambda}{\mu}\right)^{n-1} \\ &= \left(1 - \tfrac{\lambda}{\mu}\right) \tfrac{\lambda}{\mu} \textstyle\sum_{n=1}^{\infty} \tfrac{d}{dq} q^n \end{aligned}$$

where $q = \lambda/\mu$.

So,

$$\begin{aligned}
\mathsf{E}[X] &= \left(1-\tfrac{\lambda}{\mu}\right)\tfrac{\lambda}{\mu}\tfrac{d}{dq}\textstyle\sum_{n=1}^{\infty} q^n \\
&= \left(1-\tfrac{\lambda}{\mu}\right)\tfrac{\lambda}{\mu}\tfrac{d}{dq}\tfrac{q}{1-q} \\
&= \left(1-\tfrac{\lambda}{\mu}\right)\tfrac{\lambda}{\mu}\tfrac{1}{(1-q)^2} \\
&= \left(1-\tfrac{\lambda}{\mu}\right)\tfrac{\lambda}{\mu}\tfrac{1}{(1-\lambda/\mu)^2} \\
&= \tfrac{\lambda}{\mu-\lambda}
\end{aligned}$$

(b) For the M/G/1 queue, the mean queue length in equlibrium to be

$$L = \frac{2\frac{\lambda}{\mu} + \lambda^2\sigma^2 - \frac{\lambda^2}{\mu^2}}{2\left(1-\frac{\lambda}{\mu}\right)}$$

where σ^2 is the variance of the service time distribution.
In the M/M/1 queue, service times are exponential with rate μ. Hence, they have mean $1/\mu$ and variance $1/\mu^2$. So, L becomes

$$L = \frac{2\frac{\lambda}{\mu} + \lambda^2\frac{1}{\mu^2} - \frac{\lambda^2}{\mu^2}}{2\left(1-\frac{\lambda}{\mu}\right)} = \frac{2\frac{\lambda}{\mu}}{2\left(1-\frac{\lambda}{\mu}\right)} = \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu-\lambda}$$

Yeah!

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8. Let W be a waiting time of a customer arriving to the M/M/1 queue in equilibrium. Then $\mathsf{E}[W|N=0]=0$, $\mathsf{E}[W|N=1]=1/\mu$, $\dots$ $\mathsf{E}[W|N=n]=n/\mu$.

So,

$$\begin{aligned}
\mathsf{E}[W] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[W|N=n] \cdot \pi_n \\
&= \textstyle\sum_{n=0}^{\infty} \frac{n}{\mu} \cdot \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \\
&= \textstyle\sum_{n=1}^{\infty} \frac{n}{\mu} \cdot \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \\
&= \textstyle\frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty} nq^{n-1} \\
&= \textstyle\frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty} \frac{d}{dq}q^n \\
&= \textstyle\frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) \\
&= \textstyle\frac{1}{\mu}\frac{\lambda}{\mu}\frac{1}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu(\mu-\lambda)}
\end{aligned}$$

where $q = \lambda/\mu$.

For the M/M/2 system,

$$\mathsf{E}[W|N=0] = 0$$

$$\mathsf{E}[W|N=1] = 0$$

$$\mathsf{E}[W|N=2] = \frac{1}{2\mu}$$

$$\mathsf{E}[W|N=3] = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{2}{2\mu}$$

This last one used the lack of memory of the exponential. In this scenario, there are customers at both of the two servers and then another customer in line in front of our customer. We will have an expected wait of $1/2\mu$ for a customer to leave the system and which time the customer ahead of us in the line steps up for service. The other customer (one of the two originally being served), still has an exponential amount of time to go, so we will have another expected wait of $1/2\mu$ until another customer leaves and we can step up to the server.

Continuing in this manner,

$$\mathsf{E}[W|N=4] = \frac{3}{2\mu}$$

$$\vdots$$

$$\mathsf{E}[W|N=n] = \frac{n-1}{2\mu}$$

So,

$$\begin{aligned}
\mathsf{E}[W] &= \textstyle\sum_{n=0}^{\infty} \mathsf{E}[W|N=n] \cdot \pi_n \\
\mathsf{E}[W] &= \textstyle\sum_{n=2}^{\infty} \mathsf{E}[W|N=n] \cdot \pi_n.
\end{aligned}$$

We know from Problem 1 that, for the M/M/2 queue,

$$\pi_n = \frac{1}{2^{n-1}} \left(\frac{\lambda}{\mu}\right)^n \frac{2\mu - \lambda}{2\mu + \lambda}, \qquad n = 1, 2, \ldots$$

So

$$\begin{aligned} \mathsf{E}[W] &= \textstyle\sum_{n=2}^{\infty} \mathsf{E}[W|N = n] \cdot \pi_n \\ &= \textstyle\sum_{n=2}^{\infty} \frac{n-1}{2\mu} \frac{2\mu-\lambda}{2\mu+\lambda} \frac{1}{2^{n-1}} \left(\frac{\lambda}{\mu}\right)^n \\ &= \cdots \\ &= \frac{\lambda^2}{\mu(2\mu-\lambda)(2\mu+\lambda)} \end{aligned}$$

For the M/M/2 with $\lambda = 2$ and $\mu = 1.2$, the expected waiting time for a customer to get service is

$$\frac{\lambda^2}{\mu(2\mu - \lambda)(2\mu + \lambda)} = \frac{125}{66} \approx 1.8939$$

units of time.

For the M/M/1 with $\lambda = 1$ and $\mu = 1.2$, the expected waiting time for a customer to get service is

$$\frac{\lambda}{\mu(\mu - \lambda)} = \frac{25}{6} \approx 4.16667$$

units of time.

Even though the arrival rate per server is the same for both systems, the M/M/2 has two servers working so people are getting through faster!

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9. Recall that for $X \sim \Gamma(\alpha, \beta)$, $\mathsf{E}[X] = \alpha/\beta$ and $Var[X] = \alpha/\beta^2$.

   In queueing theory, we used μ to denote the lifetime or service rate, not mean. So, in this case, letting S_i denote a service time, we have

$$\frac{1}{\mu} = \mathsf{E}[S_i] = 2/\nu, \qquad \sigma^2 = 2/\nu^2.$$

   (a)

$$\begin{aligned} P(\text{has to wait for service}) &= P(\text{people in queue}) \\ &= 1 - P(\text{no one in queue}) \\ &= 1 - \pi_0 \\ &= 1 - (1 - \tfrac{\lambda}{\mu}) = \tfrac{\lambda}{\mu} = \tfrac{2\lambda}{\nu} \end{aligned}$$

(b)

$$\begin{aligned} L &= \frac{2\frac{\lambda}{\mu}+\lambda^2\sigma^2-\frac{\lambda^2}{\mu^2}}{2\left(1-\frac{\lambda}{\mu}\right)} \\ &= \frac{\frac{4\lambda}{\nu}+\frac{2\lambda^2}{\nu^2}-\frac{4\lambda^2}{\nu^2}}{2\left(1-\frac{2\lambda}{\nu}\right)} \\ &= \frac{\frac{2\lambda}{\nu}-\frac{\lambda^2}{\nu^2}}{1-\frac{2\lambda}{\nu}} \end{aligned}$$

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10. Let S be the service time of this typical customer. Let N be the number of customers that arrive during this service time. If S was fixed as s time units, then $N \sim Poisson(\lambda s)$. Hence,

$$\begin{aligned} P(N=n) &= \int_0^\infty P(N=n|S=s)\,\mu e^{-\mu s}\,ds \\ &= \int_0^\infty \frac{e^{-\lambda s}(\lambda s)^n}{n!}\,\mu e^{-\mu s}\,ds \\ &= \frac{\lambda^n\mu}{n!}\int_0^\infty s^n\,e^{-(\lambda+\mu)s}\,ds \end{aligned}$$

The integral is now looking like that of a gamma pdf with $\alpha = n+1$ and $\beta = \lambda+\mu$. We make this correspondence exact by putting in the appropriate constants

$$\begin{aligned} P(N=n) &= \frac{\lambda^n\mu}{n!}\int_0^\infty s^n\,e^{-(\lambda+\mu)s}\,ds \\ &= \frac{\lambda^n\mu}{n!}\frac{\Gamma(n+1)}{(\lambda+\mu)^{n+1}}\int_0^\infty \frac{1}{\Gamma(n+1)}(\lambda+\mu)^{n+1}s^n\,e^{-(\lambda+\mu)s}\,ds \\ &= \frac{\lambda^n\mu}{n!}\frac{\Gamma(n+1)}{(\lambda+\mu)^{n+1}}\cdot 1 \\ &= \frac{\lambda^n\mu}{n!}\frac{n!}{(\lambda+\mu)^{n+1}} = \frac{\lambda^n\mu}{(\lambda+\mu)^{n+1}} \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

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11. This is a Little's Law problem with all of the weird notation that goes with it.

Recall Little's Law:

$$L = \lambda W$$

where L is the expected number of customers in the system and W is the expected sojourn time of a typical customer. (Note that these are not random variables– they are expectations of random variables. Also note that W is being used for an expected sojourn time and not a waiting time.)

To do this problem, let W_0 be the expected waiting time for a person in the system. Since W is the expected sojourn time, we have

$$W = W_0 + \frac{1}{\mu}.$$

(i.e. We add on the customer's expected service time.)

We then need to show a variant of Little's Law $L_0 = \lambda W_0$ for the M/M/s queue. (This is true for all stable queueing models but the general proof is not easy and I would probably stick to verifing it for the M/M/s queue.)

(This is too much work... don't worry about this problem for the final!)

Finally, we have

$$L = \lambda W = \lambda \left(W_0 + \frac{1}{\mu} \right) = \lambda W_0 + \frac{\lambda}{\mu} = L_0 + \frac{\lambda}{\mu}.$$

So, the "question mark" is λ/μ.

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12. Suppose we wish to simulate values from a distribution with pdf f. To run the accept-reject algorithm, one must find a function g such that $g(x) \geq f(x)$ for all x in the support of f. One must be able to integrate g to get

$$c := \int g(x)\, dx < \infty.$$

(The integral is over the support of f.)

Define $h(x) = g(x)/c$. Note that h is a pdf.

The accept-reject algorithm is then run as follows.

[1] Simulate Y from the distribution with pdf h.

[2] Simulate $U \sim unif(0, 1)$.

[3] If

$$U \leq \frac{f(Y)}{g(Y)}$$

accept Y as a draw from f. Otherwise, discard Y and U and start over with Step 1.

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13. The Metropolis-Hastings algorithm is used to simulate values with a given pdf f. Because it involves creating a Markov chain with stationary distribution f, we usually call the "target pdf" π instead of f.

We need to choose a candidiate transition density $q(x, y)$. (The choice is arbitrary though some choices are better than others!)

Start a Makov chain $\{X_n\}$ at some arbitrary value at time 0. The evolution of the chain is as follows.

Suppose that $X_n = x$.

$\boxed{1}$ Draw a value Y from the density $q(x, y)$. Suppose that $Y = y$.

$\boxed{2}$ Draw a value $U \sim unif(0, 1)$.

$\boxed{3}$ Accept a move from x to y, and set $X_{n+1} = y$, if

$$U \leq \alpha(x, y) := \min\left\{1, \frac{\pi(y)q(y, x)}{\pi(x)q(x, y)}\right\}.$$

Otherwise, set $X_{n+1} = x$.
Return to Step 1.

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14. (a) π has detailed balance with respect to Q if

$$\pi_i q_{ij} = \pi_j q_{ji}$$

for all i, j in the state space.

(b) Suppose that

$$\pi_i q_{ij} = \pi_j q_{ji} \text{ for all } i, j$$

Sum both sides over i to get

$$\sum_i \pi_i q_{ij} = \sum_i \pi_j q_{ji} = \pi_j \sum_i q_{ji} = 0$$

since the last sum is the sum of the elements of the jth row of a generator matrix.
So, we have shown that

$$\sum_i \pi_i q_{ij} = 0 \text{ for all } j$$

The left hand side is the jth entry of the vector-matrix equation

$$\pi Q = 0.$$

So, we have verified that $\pi Q = 0$ which is a condition for stationarity for π.

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15. Um, I left out the rates. It should have said that the operating time is exponential with rate λ and the repair time is exponential with rate μ. The generator matrix is

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

The Kolmogorov forward equation is $P'(t) = P(t)Q$. The ijth entry is

$$p'_{ij}(t) = \sum_k p_{ik}(t) q_{kj}.$$

We wish to find $p_{00}(t)$. The Kolmogorov forward equation gives us that

$$p'_{00}(t) = -\lambda p_{00}(t) + \mu p_{01}(t).$$

Note that $p_{01}(t) = 1 - p_{00}(t)$. Thus, we have

$$p'_{00}(t) + (\lambda + \mu) p_{00}(t) = \mu$$

To solve this (undergrads will not have to solve differential equations on the final), multiply both sides by the integrating factor $e^{(\lambda+\mu)t}$. We then get

$$\frac{d}{dt}\left[e^{(\lambda+\mu)t} p_{00}(t)\right] = \mu e^{(\lambda+\mu)t}.$$

Integrate both sides with respect to t to get

$$e^{(\lambda+\mu)t} p_{00}(t) = \frac{\mu}{\lambda+\mu} e^{(\lambda+\mu)t} + c.$$

So, we have

$$p_{00}(t) = \frac{\mu}{\lambda+\mu} + c e^{-(\lambda+\mu)t}$$

Using the initial condition $p_{00}(0) = 1$ gives us that

$$c = 1 - \frac{\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}.$$

So, the final answer is

$$\boxed{p_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}}$$