APPM/Math 4/5520

Solutions to Final Exam Review Problems, 1-19

1.

$$\begin{aligned} \mathsf{E}[I_{\{X_1>3\}}] &= 0 \cdot P(I_{\{X_1>3\}}=0) + 1 \cdot P(I_{\{X_1>3\}}=1) \\ &= P(I_{\{X_1>3\}}=1) = P(X_1>3) = e^{-3\lambda} \end{aligned}$$

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2. (a) S is sufficient for θ if, given S, the joint distribution of $X_1, X_2, \dots, X_n$ no longer depends on θ. Intuitively then, if your goal is to estimate θ or a function of θ, once you have a sufficient statistic you no longer need the entire sample.

   (b) S is complete for θ if, given any function g such that $\mathsf{E}[g(S)] = 0$ for all θ in the parameter space, we must have that $g(S) = 0$ with probability 1. We like completeness because it means that there is only one function of S that is unbiased for what we are estimating.

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3. First note that $X_1, X_2, \dots, X_n \stackrel{iid}{\sim} Poisson(\lambda)$ implies that $S = \sum X_i$ has a negative binomial distribution. It is the first of the two negative binomial distributions found on your table of distributions. The r parameter becomes n here. (You can see this using moment generating functions!)

   In order to show that S is sufficient by the definition, we must show that the conditional distribution

   $$f_{X_1,X_2,\dots,X_n|S}(x_1, x_2, \dots, x_n|s)$$

   does not depend on p. Since the random variables involved here are discrete , this is the same thing as

   $$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n | S = s)$$

   which is

   $$\frac{P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n, S = s)}{P(S = s)} \tag{1}$$

   Now since $S = \sum X_i$, if the fixed value s isn't equal to the sum $\sum x_i$, the numerator of (1) (and therefore all of (1)) is zero.

   On the other hand, if we assume that $s = \sum x_i$,

   $$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n, S = s) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

   much in the same way that $P(X_1 = 4, X_2 = 1, X_1 + X_2 = 5) = P(X_1 = 4, X_2 = 1)$, for example.

   Therefore,

   $$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n | S = s) = \frac{P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n, S = s)}{P(S = s)}$$

$$= \frac{P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}{P(S = s)}$$

$$\stackrel{indep}{=} \frac{P(X_1 = x_1) \cdot P(X_2 = x_2) \cdots P(X_n = x_n)}{P(S = s)}$$

$$= \frac{p(1-p)^{x_1} I_{\{0,1,2,\ldots\}}(x_1) \cdots p(1-p)^{x_2} I_{\{0,1,2,\ldots\}}(x_n)}{\binom{n+s-1}{s} p^n (1-p)^s I_{\{0,1,2,\ldots\}}(s)}$$

The p's in the numerator multply together into p^n and the $(1-p)^{x_i}$ become $(1-p)^{\sum x_i} = (1-p)^s$. Thus, all p's cancel in the overall expression.

In either case, ($s \neq \sum x_i$ or $s = \sum x_i$), the conditional desnity for $X_1, X_2, \ldots, X_n$ given S does not depend on the parameter p. Therefore $S = \sum X_i$ is sufficient for this geometric distribution!

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4. Suppose that $X_1, X_2, \ldots, X_n$ has joint pdf $f(\vec{x}; \theta)$ for some parameter (or vector of parameters) θ. The statistic S is siad to be "complete" for this distribution if, for any function g such that $\mathsf{E}[g(S)] = 0$, we must have $g(S) = 0$ with probability 1.

   Complete statistics are important in our search for UMVUEs for some $\tau(\theta)$ because it gives us that there is only one function of S that is unbiased for $\tau(\theta)$. Indeed, if there were two: $g_1(S)$ and $g_2(S)$, then we would have

$$\mathsf{E}[g_1(S) - g_2(S)] = \mathsf{E}[g_1(S)] - \mathsf{E}[g_2(S)] = \tau(\theta) - \tau(\theta) = 0.$$

   By completeness of S then, we are forced to have $g_1(S) - g_1(S) = 0$ with probability 1 which gives us that $g_1(S) = g_1(S)$ with probability 1.

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5. The pdf is

$$f(x; \beta) = \frac{1}{2} \beta^3 x^2 e^{-\beta x} \, I_{(0,\infty)}(x).$$

   The joint pdf is

$$\begin{aligned} f(\vec{x}; \beta) &= \tfrac{1}{2^n} \beta^{3n} \textstyle\prod_{i=1}^n x_i^2 e^{-\beta \sum_{i=1}^n x_i} \, \prod_{i=1}^n I_{(0,\infty)}(x_i) \\ &= \tfrac{1}{2^n} \beta^{3n} \cdot \textstyle\prod_{i=1}^n x_i^2 I_{(0,\infty)}(x_i) \cdot \exp[\beta \sum x_i]. \end{aligned}$$

   By one-parameter exponential family, we have that

$$S = d(\vec{X}) = \sum_{i=1}^n X_i$$

is complete and sufficient for β.

To find the UMVUE for β, we need to find a function of S that is unbiased for β. We start by considering S itself.

$$\mathsf{E}[S] = \mathsf{E}[\sum X_i] = \sum \mathsf{E}[X_i] = \sum 3/\beta = 3n/\beta.$$

This is no good. We want to see β in the numerator. So, we will try $\mathsf{E}[1/S]$. In order to compute this, we need to realize that $S \sim \Gamma(3n, \beta)$.

$$\begin{aligned}
\mathsf{E}\left[\frac{1}{S}\right] &= \int_0^\infty \frac{1}{s} \cdot \frac{1}{\Gamma(3n)}\beta^{3n} s^{3n-1} e^{-\beta s}\, ds \\
&= \int_0^\infty \frac{1}{\Gamma(3n)}\beta^{3n} s^{3n-2} e^{-\beta s}\, ds \\
&= \frac{\Gamma(3n-1)}{\Gamma(3n)}\beta \int_0^\infty \frac{1}{\Gamma(3n-1)}\beta^{3n-1} s^{3n-2} e^{-\beta s}\, ds \\
&= \frac{1}{3n-1}\beta.
\end{aligned}$$

Therefore, by the Lehmann-Scheffé Theorem,

$$\hat{\beta} = \frac{3n-1}{S} = \frac{3n-1}{\sum X_i}$$

is the UMVUE for β.

The variance of this estimator is

$$\begin{aligned}
Var[\hat{\beta}] &= \mathsf{E}[\hat{\beta}^2] - \left(\mathsf{E}[\hat{\beta}]\right)^2 \\
&= \mathsf{E}[\hat{\beta}^2] - \beta^2
\end{aligned}$$

since $\hat{\beta}$ is an unbiased estimator of β.

Now,

$$\begin{aligned}
\mathsf{E}[\hat{\beta}^2] &= (3n-1)^2 \mathsf{E}\left[\frac{1}{S^2}\right] \\
&= (3n-1)^2 \int_0^\infty \frac{1}{s^2} \cdot \frac{1}{\Gamma(3n)}\beta^{3n} s^{3n-1} e^{-\beta s}\, ds \\
&= (3n-1)^2 \int_0^\infty \frac{1}{\Gamma(3n)}\beta^{3n} s^{3n-3} e^{-\beta s}\, ds \\
&= (3n-1)^2 \frac{\Gamma(3n-2)}{\Gamma(3n)}\beta^2 \int_0^\infty \frac{1}{\Gamma(3n-2)}\beta^{3n-2} s^{3n-3} e^{-\beta s}\, ds \\
&= (3n-1)^2 \frac{\Gamma(3n-2)}{\Gamma(3n)}\beta^2 = \frac{3n-1}{3n-2}\beta^2.
\end{aligned}$$

So, the variance is

$$Var[\hat{\beta}] = \frac{3n-1}{3n-2}\,\beta^2 - \beta^2 = \frac{1}{3n-2}\beta^2$$

6. The pdf is

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}\, I_{\{0,1,2,\ldots\}}(x)$$

The joint pdf is

$$\begin{aligned} f(\vec{x};\lambda) &= \tfrac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!} \cdot \prod I_{\{0,1,2,\ldots\}}(x_i) \\ &= e^{-n\lambda} \prod \tfrac{I_{\{0,1,2,\ldots\}}(x_i)}{x_i!} \cdot \exp[\ln\lambda \cdot \sum x_i] \end{aligned}$$

By one-parameter exponential family

$$S = d(\vec{X}) = \sum X_i$$

is complete and sufficient for λ.

To find the UMVUE for $\tau(\lambda) = \lambda^2$, we need to find a function of S that is unbiased for λ^2. We start by considering S itself.

$$\mathsf{E}[S] = \mathsf{E}[\sum X_i] = \sum \mathsf{E}[X_i] = n\lambda.$$

Since we really want to see λ^2, we'll now try

$$\mathsf{E}[S^2] = Var[S] + (\mathsf{E}[S])^2$$

Since $S \sim Poisson(n\lambda)$, this is

$$\mathsf{E}[S^2] = n\lambda + (n\lambda)^2 = \mathsf{E}[S] + n^2\lambda^2$$

and the UMVUE for λ^2 is

$$\widehat{\tau(\lambda)} = \frac{S^2 - S}{n^2} = \frac{(\sum X_i)^2 - \sum X_i}{n^2}.$$

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7. We want to find a function of $X_{(n)}$ that is unbiased for θ^p. Let's try

$$\begin{aligned} \mathsf{E}[X_{(n)}] &= \textstyle\int_0^\theta x \cdot \frac{n}{\theta^n} x^{n-1}\, dx \\ &= \tfrac{n}{n+1}\theta \end{aligned}$$

From that integral, we can see that we will get θ^p if we compute

$$\mathsf{E}[X_{(n)}^p] = \int_0^\theta x^p \cdot \frac{n}{\theta^n} x^{n-1}\, dx = \frac{n}{n+p}\,\theta^p.$$

Therefore, the UMVUE for $\tau(\theta) = \theta^p$ is

$$\widehat{\tau(\theta)} = \frac{n+p}{n}\, X_{(n)}^p.$$

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8. (a) The likelihood ratio based on this sample of size 1 is

$$\lambda(x_1; 0, \theta_1) = \frac{f(x_1; 0)}{f(x_1; \theta_1)} = \frac{1}{1 - \theta_1^2(x_1 - 1/2)}$$

Setting this less than or equal to k and flipping we get

$$1 - \theta_1^2(x_1 - 1/2) \geq \frac{1}{k}$$

$$-\theta_1^2(x_1 - 1/2) \geq \frac{1}{k} - 1$$

$$x_1 - 1/2 \leq -\frac{1}{\theta_1^2}\left(\frac{1}{k} - 1\right)$$

Note that, if θ_1 is negative or positive, θ_1^2 is always positive and $-\theta_1^2$ is always negative. So, the inequality direction at this point is independent of the sign of θ_1.

$$x_1 \leq -\frac{1}{\theta_1^2}\left(\frac{1}{k} - 1\right) + \frac{1}{2}$$

So, the form of the test is to reject if $X_1 \leq k_1$.
Now to find k_1...

$$\begin{aligned} \alpha &= P(\lambda(X_1; 0, \theta_1) \leq k; H_0) \\ &= P(X_1 \leq k_1; H_0) = k_1 \end{aligned}$$

That last inequality comes from the fact that when H_0 is true, $X_1 \sim unif(0, 1)$.
So, we take $k_1 = \alpha$ and the best (most powerful) test of the given simple versus simple hypotheses is to reject H_0 when $X_1 \geq \alpha$.

(b) Since the test from part (a) does not depend on the particular value of θ_1 (and, in this problem we specifically did not flip an inequality based on θ_1 being greater or less than 0, it is also uniformly most powerful for

$$H_0 : \theta = 0 \qquad H_1 : \theta \neq 0.$$

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9. We first consider the simple versus simple hypotheses

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 = \sigma_1^2$$

for some fixed $\sigma_1^2 > \sigma_0^2$.
The joint pdf is

$$f(\vec{x}; \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum x_i^2}.$$

The likelihood ratio is

$$\begin{aligned}
\lambda(\vec{x};\sigma_0^2,\sigma_1^2) &= \tfrac{f(\vec{x};\sigma_0^2)}{f(\vec{x};\sigma_1^2)} \\
&= \frac{(2\pi\sigma_0^2)^{-n/2}e^{-\frac{1}{2\sigma_0^2}\sum x_i^2}}{(2\pi\sigma_1^2)^{-n/2}e^{-\frac{1}{2\sigma_1^2}\sum x_i^2}} \\
&= (\sigma_1^2/\sigma_0^2)^{n/2}\cdot e^{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum x_i^2}
\end{aligned}$$

Setting this less than or equal to k and starting to move things, we get

$$e^{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum x_i^2} \leq (\sigma_0^2/\sigma_1^2)^{n/2}k$$

$$-\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum x_i^2 \leq \ln\left[(\sigma_0^2/\sigma_1^2)^{n/2}k\right]$$

$$\sum x_i^2 \geq \frac{\ln\left[(\sigma_0^2/\sigma_1^2)^{n/2}k\right]}{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)}$$

since $\sigma_1^2 > \sigma_0^2$.

So, the best test of

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 = \sigma_1^2$$

for some fixed $\sigma_1^2 > \sigma_0^2$ will be to reject H_0 if

$$\sum X_i^2 \geq k_1$$

where k_1 is chosen to give a size α test.

Now let's find k_1.

$$\alpha = P\left(\sum X_i^2 \geq k_1; H_0\right)$$

Since, under H_0, $X_i \sim N(0,\sigma_0^2)$ so $X_i/\sigma_0^2 \sim N(0,1)$. Squaring a $N(0,1)$ gives a χ^2 random variable. Adding independent χ^2-random variables gives another χ^2 with all the degrees of freedom added up.

So,

$$\frac{\sum_{i=1}^n X_i^2}{\sigma_0^2} = \sum_{i=1}^n \frac{X_i^2}{\sigma_0^2} = \sum_{i=1}^n \left(\frac{X_i}{\sigma_0}\right)^2 \sim \chi^2(n)$$

So,

$$\begin{aligned}
\alpha &= P\left(\sum X_i^2 \geq k_1; H_0\right) \\
&= P\left(\frac{\sum X_i^2}{\sigma_0^2} \geq k_1/\sigma_0^2; H_0\right) \\
&= P(W > k_1/\sigma_0^2)
\end{aligned}$$

where $W \sim \chi^2(n)$.

So, we have that k_1/σ_0^2 is the $\chi^2(n)$ critical value that cuts off area α to the right. Our notation for this is $\chi^2_\alpha(2n)$. So

$$k_1 = \sigma_0^2\, \chi^2_\alpha(2n).$$

So, the best test of size α of

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 = \sigma_1^2$$

for some fixed $\sigma_1^2 > \sigma_0^2$ will be to reject H_0 if

$$\sum X_i^2 \geq \sigma_0^2\, \chi^2_\alpha(2n).$$

This test does not depend on the specific chosen value of σ_1^2 (with the exception that the form of the test depends on the fact that $\sigma_1^2 > \sigma_0^2$). So, this is a UMP test of size α for

$$H_0 : \sigma^2 = \sigma_0^2 \text{ versus } H_1 : \sigma^2 > \sigma_0^2.$$

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10. The power function is

$$\begin{aligned} \gamma(\sigma^2) &= P(\text{Reject } H_0\,; \sigma^2) \\ &= P(\textstyle\sum X_i^2 \geq \sigma_0^2\, \chi^2_\alpha(2n)\,; \sigma^2) \end{aligned}$$

We are under the assumption that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(0, \sigma^2)$. We don't know the distribution of these squared, but we would if they were $N(0,1)$. ($N(0,1)$ random variables squared are $\chi^2(1)$ random variables.) Note that

$$\frac{\sum X_i^2}{\sigma^2} = \sum \left(\frac{X_i}{\sigma}\right)^2 \sim \chi^2(n)$$

So, back to the power function...

$$\begin{aligned} \gamma(\sigma^2) &= P(\textstyle\sum X_i^2 \geq \sigma_0^2\, \chi^2_\alpha(2n)\,; \sigma^2) \\ &= P\left(\frac{\sum X_i^2}{\sigma^2} \geq \frac{\sigma_0^2\, \chi^2_\alpha(2n)}{\sigma^2}\,; \sigma^2\right) \\ &= P\left(W \geq \frac{\sigma_0^2\, \chi^2_\alpha(2n)}{\sigma^2}\right) = 1 - F_W\left(\frac{\sigma_0^2\, \chi^2_\alpha(2n)}{\sigma^2}\right) \end{aligned}$$

where $W \sim \chi^2(n)$.

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11. (a) We should reject H_0 if the minimum is large. So a test based on $X_{(1)}$ should look like

"Reject H_0 if $X_{(1)} > c$"

Now find c.

$$\begin{aligned}\alpha &= P(\text{ Reject } H_0 \text{ when true }) \\ &= P(X_{(1)} > c; \theta_0) \\ &= e^{-n\theta_0 c}\end{aligned}$$

since the minimum of exponentials with rate θ_0 is exponential with rate $n\theta_0$.
So

$$c = -\frac{1}{n\theta_0}\ln(\alpha)$$

So a test of size α of the given hypotheses and based on $X_{(1)}$ is to reject H_0 if

$$X_{(1)} > -\frac{1}{n\theta_0}\ln(\alpha).$$

(b) To find the UMP test, we first consider the simple versus simple hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for some fixed $\theta_1 < \theta_0$.
The Neyman-Pearson Lemma tells us to consider the likelihood ratio

$$\lambda(\vec{x};\theta_0,\theta_1) = \frac{f(\vec{x};\theta_0)}{f(\vec{x};\theta_1)} = \cdots = \left(\frac{\theta_0}{\theta_1}\right)^n e^{-(\theta_0-\theta_1)\sum x_i}$$

and that we should reject H_0 when this is less than or equal to some k, to be determined.
Now

$$\begin{gathered}\left(\frac{\theta_0}{\theta_1}\right)^n e^{-(\theta_0-\theta_1)\sum x_i} \le k \\ \Downarrow \\ e^{-(\theta_0-\theta_1)\sum x_i} \le k_1 \\ \Downarrow \\ -(\theta_0-\theta_1)\sum x_i \le k_2 \\ \Downarrow \\ \sum x_i \ge k_3\end{gathered}$$

for some k_3. Note that the inequality flipped because $\theta_1 < \theta_0$.
To find the k_3,

$$\begin{aligned}\alpha &= P(\text{ Type I Error }) \\ &= P(\text{ Reject } H_0 \text{ when true }) \\ &= P(\textstyle\sum X_i \ge k_3; \theta_0) \\ &= P(W \ge k_3)\end{aligned}$$

where $W \sim \Gamma(n, \theta_0)$.
Since we can't get a closed form solution, we will move to express the test in terms of a χ^2 critical value. Note that $2\theta_0 W \sim \Gamma(n, 1/2) = \chi^2(2n)$. So,

$$\alpha = P(W \ge k_3) = P(\underbrace{2\theta_0 W}_{\sim\chi^2(2n)} \ge 2\theta_0 k_3)$$

implies that $2\theta_0 k_3 = \chi^2_{\alpha,2n}$.
So, the best size α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for $\theta_1 < \theta_0$ is to reject H_0 in favor of H_1 if

$$\sum_{i=1}^{n} X_i \geq \chi^2_\alpha(2n)/(2\theta_0).$$

Since this test does not depend on the particular value of θ_1 used (only on the fact that it is less than θ_0, we have that the test is UMP for $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$. That is, the UMP test is to reject H_0 if

$$\sum X_i \geq \chi^2_\alpha(2n)/(2\theta_0).$$

(c) The power functions...
For the test from part (a):

$$\begin{aligned}\gamma_{(a)}(\theta) &= P(\text{ reject } H_0 \text{ when the parameter is } \theta\,) \\ &= P\left(X_{(1)} < -\tfrac{1}{n\theta_0}\ln(1-\alpha);\theta\right)\end{aligned}$$

When the parameter is θ, $X_{(1)}$ is exponential with rate $n\theta$. So

$$\begin{aligned}\gamma_{(a)}(\theta) &= 1 - e^{n\theta\left(-\frac{1}{n\theta_0}\ln(1-\alpha)\right)} \\ &= 1-(1-\alpha)^{(-\theta/\theta_0)}\end{aligned}$$

For the test from part (b):

$$\begin{aligned}\gamma_{(b)}(\theta) &= P(\text{ reject } H_0 \text{ when the parameter is } \theta\,) \\ &= P\left(\sum X_i \geq \chi^2_\alpha(2n)/(2\theta_0);\theta\right)\end{aligned}$$

When the parameter is θ, $\sum X_i \sim \Gamma(n,\theta)$. So $2\theta \sum X_i \sim \Gamma(n, 1/2) = \chi^2(2n)$. Therefore

$$\begin{aligned}\gamma_{(b)}(\theta) &= P\left(2\theta \sum X_i \geq 2\theta\chi^2_\alpha(2n)/2\theta_0;\theta\right) \\ &= P\left(W \geq \theta\chi^2_\alpha(2n)/\theta_0\right)\end{aligned}$$

This function is one minus the cdf of a $\chi^2(2n)$ random variable evaluated at $\theta\chi^2_\alpha(2n)/\theta_0$ for a fixed n, a fixed θ_0 and regarded as a function of θ. There is no nice closed form expression for comparison to the other power function. For fixed n and θ_0, you could numerically plot the expression– when plotted along with $\gamma_{(a)}(\theta)$ you should see that $\gamma_{(b)}(\theta)$ is above $\gamma_{(a)}(\theta)$ for all values of θ.

12. To begin, we need to find any unbiased estimator. Note that, for this Poisson distribution, $P(X = 0) = e^{-\lambda}$. So, an unbiased estimator is $I_{\{X_1=0\}}$.

By one-parameter exponential family, it is easy to see that $S = \sum X_i$ is complete and sufficient for this distribution. By the Rao-Blackwell Theorem, we know that $\mathsf{E}[I_{\{X_1=0\}}|S]$ is also an unbiased estimator for $\tau(\lambda)$ and furthermore that it is a function of S. Since S is complete and sufficient, we will then have found the UMVUE.

For ease of computation, we will begin by putting in a value for S:

$$\begin{aligned}
\mathsf{E}[I_{\{X_1=0\}}|S=s] &= P(X_1=0|S=s) \\
&= \tfrac{P(X_1=0,S=s)}{P(S=s)} \\
&= \tfrac{P\left(X_1=0,\sum_{i=1}^n X_i=s\right)}{P(S=s)} \\
&= \tfrac{P\left(X_1=0,\sum_{i=2}^n X_i=s\right)}{P(S=s)} \\
&\overset{indep}{=} \tfrac{P(X_1=0)\cdot P\left(\sum_{i=2}^n X_i=s\right)}{P(S=s)}
\end{aligned}$$

Since $\sum_{i=1}^n X_i \sim Poisson(n\lambda)$ and $\sum_{i=2}^n X_i \sim Poisson((n-1)\lambda)$, this is equal to

$$\frac{e^{-\lambda}\cdot\frac{e^{-(n-1)\lambda}[(n-1)\lambda]^s}{s!}}{\frac{e^{-n\lambda}[n\lambda]^s}{s!}} = \left(\tfrac{n-1}{n}\right)^s.$$

Removing the specific value of S, we have that

$$\widehat{\tau(\lambda)} = \left(\frac{n-1}{n}\right)^S = \left(\frac{n-1}{n}\right)^{\sum_{i=1}^n X_i}.$$

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13. (a)

$$\begin{aligned}
& f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}}e^{-\frac{1}{2\theta}x^2} \\
\Rightarrow \quad & \ln f(x;\theta) = -\frac{1}{2}\ln(2\pi\theta) - \frac{1}{2\theta}x^2 \\
\Rightarrow \quad & \frac{\partial}{\partial\theta}\ln f(x;\theta) = -\frac{1}{2}\frac{1}{\theta} + \frac{1}{2\theta^2}x^2
\end{aligned}$$

So,

$$\begin{aligned}
I_1(\theta) &= \mathsf{E}\left[\left(\tfrac{\partial}{\partial\theta}\ln f(X;\theta)\right)^2\right] \\
&= \mathsf{E}\left[\left(\tfrac{1}{2\theta^2}X^2 - \tfrac{1}{2\theta}\right)^2\right] \\
&= \tfrac{1}{4\theta^4}\mathsf{E}\left[(X^2-\theta)^2\right]
\end{aligned}$$

Now $X \sim N(0,\theta) \Rightarrow X/\sqrt{\theta} \sim N(0,1) \Rightarrow X^2/\theta \sim \chi^2(1)$, so let's go back to the second to last inequality above (in computing $I_1(\theta)$), and only pull out a single θ:

$$\begin{aligned}
I_1(\theta) &= \mathsf{E}\left[\left(\tfrac{1}{2\theta^2}X^2 - \tfrac{1}{2\theta}\right)^2\right] \\
&= \tfrac{1}{4\theta^2}\mathsf{E}\left[\left(\tfrac{1}{\theta}X^2 - 1\right)^2\right] \\
&= \tfrac{1}{4\theta^2}Var(W)
\end{aligned}$$

where $W \sim \chi^2(1)$. (Since $\mathsf{E}[W] = 1$.)
So

$$I_1(\theta) = \frac{1}{4\theta^2} \cdot 2 = \frac{1}{2\theta^2}$$

and

$$I_n(\theta) \stackrel{iid}{=} n \cdot I_1(\theta) = \frac{n}{2\theta^2}.$$

(b) The MLE of θ is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^2}{n}.$$

Hence,

$$Var(\hat{\theta}) = Var\left(\frac{\sum_{i=1}^{n} X_i^2}{n}\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i^2\right) \stackrel{indep}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2) \stackrel{ident}{=} \frac{1}{n^2} \cdot n \cdot Var(X_1^2)$$

Now we could go the long way and write $Var(X_1^2) = \mathsf{E}[X_1^4] - \left(\mathsf{E}[X_1^2]\right)^2$, or we could go the short way and observe again that $X^1/\theta \sim \chi^2(1)$. Therefore

$$Var(\hat{\theta}) = \frac{1}{n} Var(X_1^2) = \frac{1}{n} Var\left(\theta \frac{X_1^2}{\theta}\right) = \frac{1}{n}\theta^2 Var\left(\frac{X_1^2}{\theta}\right) = \frac{1}{n}\theta^2 Var(W) = \frac{1}{n}\theta^2 \cdot 2$$

This is exactly the same as the CRLB for θ:

$$CRLB_\theta = \frac{\left[\frac{\partial}{\partial\theta}\theta\right]^2}{I_n(\theta)} = \frac{1}{n/(2\theta^2)} = \frac{2\theta^2}{n}.$$

Hence, the MLE is efficient!

14.

$$f(x;\theta) = \frac{e^{-\lambda}\lambda^x}{x!} = e^{\lambda} \cdot \frac{1}{x!} \cdot \exp\left[x \cdot \ln \lambda\right]$$

implies, by one-parameter exponential family, that $d(X) = X$ is complete and sufficient for λ. (Note: We have a sample of size 1, so we look at the pdf for X alone as opposed to a joint pdf of several X's.)

We are given a function of this complete and sufficient statistic, namely $(-1)^X$ which is supposed to be the UMVUE for $e^{-2\mu} = e^{-2\lambda}$. ($\mu$ denotes the mean of the distribution which is simply λ for a Poisson rate λ distribution.)

Now,

$$\begin{aligned} \mathsf{E}\left[(-1)^X\right] &= \textstyle\sum_{x=0}^{\infty}(-1)^x \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \textstyle\sum_{x=0}^{\infty} \frac{e^{-\lambda}(-\lambda)^x}{x!} \\ &= e^{-\lambda}\textstyle\sum_{x=0}^{\infty} \frac{(-\lambda)^x}{x!} \\ &= e^{-\lambda} \cdot e^{-\lambda} = e^{-2\lambda} \end{aligned}$$

as desired. Hence $(-1)^X$ is the UMVUE for $e^{-2\lambda}$.

---

15. (a)

$$\begin{aligned} f(\vec{x};\theta) &= \tfrac{(\ln\theta)^{\sum x_i}}{\theta^n \prod (x_i!)} \cdot \prod I_{\{0,1,\ldots\}}(x_1) \\ &= \tfrac{1}{\theta^n} \cdot \tfrac{1}{\prod (x_i)!} \prod I_{\{0,1,\ldots\}}(x_i) \cdot \exp\left[(\sum x_i) \cdot \ln(\ln\theta)\right] \end{aligned}$$

So, by one-parameter exponential family, we see that $S = \sum X_i$ is complete and sufficient for θ.

To find the UMVUE for $\ln\theta$, we need to find a function of $S = \sum X_i$ that is unbiased for $\ln\theta$. We start by considering S itself:

$$\mathsf{E}[S] = n \cdot \mathsf{E}[X_1] = n \cdot \ln\theta$$

since $X_1 \sim Poisson(\ln\theta)$.

Hence,

$$\overline{X}$$

is the UMVUE for $\ln\theta$.

(b) To find the UMVUE for $(\ln\theta)^2$, we need to find a function of $S = \sum X_i$ that is unbiased for $(\ln\theta)^2$. We start by considering S^2:

$$\mathsf{E}[S^2] = \mathsf{E}[(\sum X_i)^2] = Var(\sum X_i) + \left(\mathsf{E}[\sum X_i]\right)^2$$

Since the X's are iid, $Var(\sum X_i) = \sum Var(X_i) = nVar(X_1) = n\ln\theta$ and we have that

$$\mathsf{E}[S^2] = n \cdot \ln\theta + n^2(\ln\theta)^2$$

Since $\mathsf{E}[S] = n\ln\theta$, we have that

$$\frac{S^2 - S}{n^2} = \frac{(\sum X_i)^2 - \sum X_i}{n^2}$$

is an unbiased ()for $(\ln\theta)^2$ function of the complete and sufficient statistic S and hence is the UMVUE for $(\ln\theta)^2$.

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16. Consider first the simple versus simple hypotheses:

$$H_0 : \theta = \theta_0 \qquad H_1 : \theta = \theta_1$$

for some $\theta_1 < \theta_0$. The ratio for the Neyman-Pearson test is

$$\lambda(\vec{x};\theta_0,\theta_1) = \frac{\frac{1}{\theta_0^n} I_{(0,\theta_0)}(x_{(n)}) \cdot I_{(0,x_{(n)})}(x_{(1)})}{\frac{1}{\theta_1^n} I_{(0,\theta_1)}(x_{(n)}) \cdot I_{(0,x_{(n)})}(x_{(1)})} = \left(\frac{\theta_1}{\theta_0}\right)^n \frac{I_{(0,\theta_0)}(x_{(n)})}{I_{(0,\theta_1)}(x_{(n)})} \overset{set}{\leq} k$$

The k should be something non-negative since λ is a ratio of pdfs and therefore is always non-negative. Note that if the indicator in the numerator is zero if $x_{(n)} > \theta_0$. In this case, we absolutely know that H_0 is not true since it states that all values in the sample will be between 0 and θ_0. This is reflected in the fact that $x_{(n)} > \theta_0 \Rightarrow \lambda = 0$ which is less than or equal to any valid k, so we will always reject.

On the other hand, if the indicator in the denominator is zero, this means that $x_{(n)} > \theta_1$. The N-P ratio λ becomes infinite (in a sense) which makes it NOT less than or equal to any cut-off k, so we would never reject H_0. This makes sense because $x_{(n)} > \theta_1$ implies that H_1 could not possibly be true since it says that all values in the sample are between 0 and θ_1.

All of these comments aside, this test is garbage if $x(n)$ is greater than both θ_0 and θ_1 since, in hypothesis testing, the assumption is that one of the two hypotheses is true. Since $\theta_1 < \theta_0$, and the sample came from either the $unif(0, \theta_0)$ or $unif(0, \theta_1)$ distribution, we must have that $x_{(n)} < \theta_0$, and so the indicator in the numerator is one. Thus, we have

$$\left(\frac{\theta_1}{\theta_0}\right)^n \frac{1}{I_{(0,\theta_1)}(x_{(n)})} \leq k$$

$$\Rightarrow \qquad \frac{1}{I_{(0,\theta_1)}(x_{(n)})} \leq \left(\frac{\theta_0}{\theta_1}\right)^n k$$

$$\Rightarrow I_{(0,\theta_1)}(x_{(n)}) \geq k_1$$

Now the indictor will be "large" (ie: 1) if $x_{(n)}$ is small, so this is equivalent to

$$X_{(n)} \leq k_2$$

for some k_2 such that

$$P(X_{(n)} \leq k_2; \theta_0) = \alpha$$

ie:

$$\left(\frac{k_2}{\theta_0}\right)^n = \alpha$$

$$\Rightarrow \qquad k_2 = \theta_0 \alpha^{1/n}$$

So, the UMP test of

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta = \theta_1$$

is to reject H_0 if $X_{(n)} \leq \theta_0 \alpha^{1/n}$. Since this test does not involve θ_1 (only that $\theta_1 < \theta_0$), it is UMP for

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta < \theta_0$$

Finally, the composite null hypothesis will only chage the way the level of significance is defined

$$\begin{aligned} \alpha &= \max_{\theta \geq \theta_0} P(X_{(n)} \leq k_2; \theta) \\ &= \max_{\theta \geq \theta_0} \left(\tfrac{k_2}{\theta}\right)^n = \left(\tfrac{k_2}{\theta_0}\right)^n \\ &\Rightarrow \qquad k_2 = \theta_0 \alpha^{1/n} \end{aligned}$$

So, a UMP test of size α of

$$H_0 : \theta \leq \theta_0 \qquad \text{versus} \qquad H_1 : \theta < \theta_0$$

is to reject H_0 if $X_{(n)} \leq \theta_0 \alpha^{1/n}$.

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17. The pdf is

$$f(x;\theta_1,\theta_2) = \frac{1}{\theta_2-\theta_1}\, I_{[\theta_1,\theta_2]}(x).$$

The joint pdf is

$$\begin{aligned} f(\vec{x};\theta_1,\theta_2) &= \tfrac{1}{\theta_2-\theta_1}^{n} \textstyle\prod_{i=1}^{n} I_{[\theta_1,\theta_2]}(x_i) \\ &= \tfrac{1}{\theta_2-\theta_1}^{n} I_{[\theta_1,\theta_2]}(x_{(1)}) I_{[\theta_1,\theta_2]}(x_{(n)}) \end{aligned}$$

By the Factorization Criterion for sufficiency, we see that

$$S = (X_{(1)}, X_{(n)})$$

is sufficient for this distribution.

---

18. (a) First note that, when the parameter is in the indicator like this, the exponential family factorization for find a complete and sufficient statistic will never work. That factorization is about complete separation of the x's and θ ($a(\theta)$, $b(\vec{x})$, $c(\theta)$, $d(\vec{x})$) but they are stuck together in the indicator.

First, we need to find a sufficient statistic. We'll use the Factorization Criterion:

$$f(\vec{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \cdots = e^{n\theta-\sum x_i} I_{(\theta,\infty)}(x_{(1)}) = \underbrace{e^{-\sum x_i}}_{h(\vec{x})} \underbrace{e^{n\theta} I_{(\theta,\infty)}(x_{(1)})}_{g(s(\vec{x});\theta)}$$

Thus, we see that $S = X_{(1)}$ is sufficient for θ.

To show that S is complete, we need to find the pdf for the minimum. I am running out of time and need to get these solutions posted, so I am omitting the details, but the pdf for the minimum is

$$f_{X_{(1)}}(x) = ne^{n(\theta-x)} I_{(\theta,\infty)}(x)$$

To show completeness, assume that g is any function such that $\mathsf{E}[g(X_{(1)})] = 0$ for all θ. Then

$$0 = \mathsf{E}[g(X_{(1)})] = \textstyle\int_\theta^\infty g(x)\, n\, e^{n(\theta-x)}\, dx = ne^{n\theta} \int_\theta^\infty g(x)\, e^{-nx}\, dx$$

for all θ. This implies that

$$\int_\theta^\infty g(x)\, e^{-nx}\, dx = 0$$

or, equivalently,

$$-\int_\infty^\theta g(x)\, e^{-nx}\, dx = 0$$

and thus

$$\int_\infty^\theta g(x)\, e^{-nx}\, dx = 0$$

for all θ.

Taking the derivative of both sides with respect to θ gives

$$g(\theta)e^{-n\theta} = 0$$

for all θ. Since $e^{-n\theta} \neq 0$, we get that $g(\theta)$ must be zero for all θ. Thus, $g(X_{(1)}) = 0$ and we have that $S = X_{(1)}$ is complete for θ.

(b) We need to find a function of $X_{(1)}$ that is unbiased for θ. We consider $X_{(1)}$ itself.

$$\begin{aligned}
\mathsf{E}[X_{(1)}] &= \textstyle\int_{-\infty}^{\infty} x f_{X_{(1)}}(x)\,dx \\
&= \textstyle\int_{\theta}^{\infty} xne^{n(\theta-x)}\,dx \\
&= ne^{n\theta}\textstyle\int_{\theta}^{\infty} xe^{-nx}\,dx \\
&= e^{n\theta}[\theta e^{-n\theta} + \frac{1}{n}e^{-n\theta}] \\
&= \theta + \frac{1}{n}
\end{aligned}$$

So, $\widehat{\theta} = X_{(1)} - 1/n$.

---

19. The joint pdf is

$$f(\vec{x};\theta) = \theta^n \left[\prod_{i=1}^{n}(1-x_i)\right]^{\theta-1} \prod_{i=1}^{n} I_{(0,1)}(x_i).$$

A likelihood is

$$L(\theta) = \theta^n \left[\prod_{i=1}^{n}(1-x_i)\right]^{\theta-1}.$$

The MLE (work not shown) is

$$\widehat{\theta} = \frac{-n}{\sum \ln(1-X_i)}.$$

The restricted MLE is $\widehat{\theta}_0 = 1$.

The GLR is

$$\lambda(\vec{X}) = \frac{L(\hat{\theta}_0)}{L(\hat{\theta})}.$$

Note that $L(\widehat{\theta}_0) = L(1) = 1$. Thus, the GLR is

$$\lambda(\vec{X}) = \left(\frac{\sum \ln(1-X_i)}{-n}\right)^n \left[\prod_{i=1}^{n}(1-x_i)\right]^{\frac{n}{\sum \ln(1-X_i)}+1}.$$

The form of the GLRT is to reject H_0 in favor of H_1 if

$$\left(\frac{\sum \ln(1-X_i)}{-n}\right)^n \left[\prod_{i=1}^{n}(1-x_i)\right]^{\frac{n}{\sum \ln(1-X_i)}+1} \leq k$$

where k is to be determined so that $P(\lambda(\vec{X}) \leq k; 1) = \alpha$.

---

20. The restricted MLE is $\hat{\mu}_0 = \mu_0$ (Here, μ_0 is notation for the constant that is given in the setup of the hypotheses and $\hat{\mu}_0$ is notation for the MLE estimator for μ restricted to when H_0 is true.)

    The unrestricted MLE is $\overline{X}$.

    Therefore, the GLR is

$$\lambda(\vec{X}) = \frac{L(\hat{\mu}_0)}{L(\hat{\mu})} = \frac{(2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(X_i-\mu_0)^2}}{(2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(X_i-\overline{X})^2}} = e^{-\frac{1}{2\sigma^2}\sum[(X_i-\mu_0)^2-(X_i-\overline{X})^2]}$$

$$= e^{-\frac{1}{2\sigma^2}\sum[(X_i-\mu_0)^2-(X_i-\overline{X})^2]}$$

    (Note that the σ^2's in the front of the e's could cancel because, in this problem, σ^2 is fixed and known.)

    After a bit of simplification, this can be expressed as

$$\lambda(\vec{x}) = \exp[-n(\overline{x} - \mu_0)^2/2\sigma^2]$$

    We reject H_0 if $\lambda(\vec{x}) \leq k$ which is equivalent to

$$\frac{-n(\overline{x} - \mu_0)^2}{2\sigma^2} \leq k_1$$

$$\Rightarrow \quad \frac{n(\overline{x} - \mu_0)^2}{\sigma^2} \geq k_2$$

$$\Rightarrow \quad \left(\frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}\right)^2 \geq k_2 \tag{2}$$

    We now could choose to take the square root of both sides which would give us

$$\frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \geq k_3 \qquad \text{or} \qquad \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \leq -k_3, \tag{3}$$

    (where $k_3 = \sqrt{k_2}$) or we could leave things in the form of (2). Either answer would be correct.

    <u>Case 1</u>: Leave things in the form of (2).

    Here, we choose k_2 such that

$$P\left(\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right)^2 \geq k_2; \mu_0\right) = \alpha$$

    When $\mu = \mu_0$, $\overline{X} \sim N(\mu_0, \sigma^2/n) \Rightarrow (\overline{X} - \mu_0)/(\sigma/\sqrt{n}) \sim N(0,1)$ *Rightarrow* $[(\overline{X} - \mu_0)/(\sigma/\sqrt{n})]^2 \sim \chi^2(1) \Rightarrow k_2 = \chi^2_\alpha(1)$.

    So, the GLRT of size α is to reject H_0 if

$$\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}\right)^2 \geq \chi^2_\alpha(1).$$

Alternatively, we have....

Case 2: Leave things in the form of (3).

Here we find k_3 such that

$$P\left(\frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \geq k_3 \;\text{ or }\; \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \leq -k_3; \mu_0\right) = \alpha$$

Since $\mu = \mu_0$, this is equivalent to

$$P(Z \geq k_3 \;\text{ or }\; Z \leq -k_3) = \alpha$$

$$\Rightarrow \qquad k_3 = z_{\alpha/2}.$$

So, the GLRT of size α is to reject H_0 if

$$\frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \geq z_{\alpha/2} \;\text{ or }\; \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \leq -z_{\alpha/2}$$

If you used "Case 2", the GLRT is exactly the "common sense" two-tailed test from an earlier part of the course. Using "Case 1", we get the chi-squared test exactly without having to resort to asymptotics.

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21. The joint pdf for X and Y is

$$f_{X,Y}(x,y) = \binom{n_1}{x} p_1^x (1-p_1)^{n_1-x} \cdot \binom{n_2}{y} p_2^y (1-p_2)^{n_2-y}$$

(a) The resctricted MLE:

We assume that $p_1 = p_2$ and denote the common value denoted simply by p. Then

$$f_{X,Y}(x,y) = \binom{n_1}{x}\binom{n_2}{y} p^{x+y}(1-p)^{n_1+n_2-(x+y)}$$

$$\Rightarrow \qquad L(p) = p^{x+y}(1-p)^{n_1+n_2-(x+y)}$$

$$\ln L(p) = (x+y)\ln p + (n_1+n_2-(x+y))\ln(1-p)$$

$$\frac{\partial}{\partial p}\ln L(p) = \frac{x+y}{p} - \frac{n_1+n_2-(x+y)}{1-p} \stackrel{set}{=} 0$$

$$\Rightarrow \qquad \hat{p}_0 = \frac{x+y}{n_1+n_2}$$

where $\hat{p}_0$ denotes the restricted MLE for p.

The unrestricted MLE's for p_1 and p_2:

Recall that the joint pdf for X and Y is

$$f_{X,Y}(x,y) = \binom{n_1}{x} p_1^x (1-p_1)^{n_1-x} \cdot \binom{n_2}{y} p_2^y (1-p_2)^{n_2-y}$$

So, a likelihood function is

$$L(p_1, p_2) = p_1^x (1-p_1)^{n_1-x} \cdot p_2^y (1-p_2)^{n_2-y}$$

and the log is

$$\ln L(p_1,p_2) = x \ln p_1 + (n_1 - x)\ln(1-p_1) + y \cdot \ln p_2 + (n_2 - y)\ln(1-p_2)$$

$$\begin{aligned}
\tfrac{\partial}{\partial p_1} \ln L(p_1,p_2) &= \tfrac{x}{p_1} - \tfrac{n_1-x}{1-p_1} \overset{set}{=} 0 \\
\tfrac{\partial}{\partial p_2} \ln L(p_1,p_2) &= \tfrac{y}{p_2} - \tfrac{n_2-y}{1-p_2} \overset{set}{=} 0 \\
\Rightarrow \qquad & \hat{p}_1 = \frac{x}{n_1}, \;\; \hat{p}_2 = \frac{y}{n_2}
\end{aligned}$$

So, the GLR is

$$\lambda(\vec{x}) = \frac{\left(\frac{x+y}{n_1+n_2}\right)^{x+y}\left(1-\frac{x+y}{n_1+n_2}\right)^{n_1+n_2-(x+y)}}{\left(\frac{x}{n_1}\right)^{x}(1-\left(\frac{x}{n_1}\right))^{n_1-x}\cdot\left(\frac{y}{n_2}\right)^{y}(1-\left(\frac{y}{n_2}\right))^{n_2-y}}$$

(b) The approximate large sample GLRT of size α is to reject H_0 if

$$-2\ln\lambda(\vec{X}) \geq \chi^2_\alpha(2)$$

(2 is the number of parameters restricted in the null hypothesis.)