Remember to write your name! You are allowed to use a calculator. **You are not allowed to use the textbook or your notes or your neighbor**. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise. You may cite any theorem from Atkinson or from the lectures unless explicitly stated otherwise.

You must do the first problem. You must pick **only** two of the remaining problems. Each problem is 15 points; there are 45 points total.

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**1. Quadrature** Let

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{i=0}^{n} w_{i,n} f(x_{i,n})$$

for $n = 1, \ldots, \infty$ be a family of quadratures. For parts (a) and (b), assume that the quadrature with $n + 1$ nodes exactly integrates polynomials up to degree at least $n$.

(a) Prove that if all the weights are non-negative $w_{i,n} \geq 0$ and $f \in C[a, b]$ then the quadratures will converge to the integral as $n \to \infty$. (Don't just quote the theorem; present the proof.)

(b) Suppose that the function values $f(x_{i,n})$ are corrupted with errors $\epsilon_{i,n}$ (e.g. due to roundoff), and consider the approximation

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{i=0}^{n} w_{i,n}(f(x_{i,n}) + \epsilon_{i,n}).$$

Prove that if the weights are non-negative and the errors are uniformly bounded $|\epsilon_{i,n}| \leq \epsilon$ then the quadrature based on the corrupted values will converge to within an error $\leq (b-a)\epsilon$ as $n \to \infty$. Notice that this implies there's a point of diminishing returns where it is no longer valuable to increase $n$, e.g. once the error has already reached machine precision (No work required by you for this sentence).

(c) Suppose you're using the equispaced composite trapezoid rule, and the function values are corrupted with errors as in (b). Prove that as $n \to \infty$ the quadrature will converge to within an error $\leq (b - a)\epsilon$.

**2. Linear Systems**

(a) Prove that the Jacobi iteration will converge for any strictly-diagonally-dominant linear system of equations. (Prove the theorem, don't just quote it.)

(b) The Gauss-Seidel iteration has the pseudocode form

- until convergence, do (this is the 'outer loop')
  - for $i = 1 : n$ (this is the 'inner loop')
    - Solve equation $i$ for $x_i$ with all values $x_j$, $j \neq i$ held fixed
    - Set $x_i$ equal to the solution found in the previous step

Prove the following: If the linear system is symmetric positive definite, then the inner loop can be computed in any order (e.g. $n, n-1, \ldots, 1$ instead of $1, \ldots, n$), and the iteration will still converge.

**3. Rootfinding/Nonlinear Equations** Suppose that $f(x)$ is globally Lipschitz continuous with $|f(x) - f(y)| \leq 2|x - y| \; \forall \, x, y \in \mathbb{R}$.

(a) Show that for sufficiently small $\delta_t > 0$, there is a unique solution $x$ to $x = y + \delta_t f(x)$ for any $y \in \mathbb{R}$.

(b) Suppose that $f$ has a continuous derivative, and assume (don't prove) that the solution in part (a) is a smooth function of $y$. Suppose also that $f(\alpha) = 0$ for some $\alpha \in \mathbb{R}$. Under what conditions on $\delta_t$ and $f'(\alpha)$ will the iteration

$$x_{k+1} = x_k + \delta_t f(x_{k+1})$$

converge to $\alpha$? If the iteration is convergent then you can say 'for close enough initial conditions the iteration will converge' without specifying precisely how close is 'close enough.'

**4. Interpolation**

(a) The set of splines of fixed degree with fixed nodes is a vector space. Suppose that the splines are piecewise-cubic on each subinterval, and there are $n$ subintervals ($n + 1$ nodes). What is the dimension of the vector space?

(b) The number of interpolation conditions ($n + 1$) is sometimes less than the dimension of the vector space. In the cubic example above, give three different kinds of extra conditions that can be imposed to make the dimension of the vector space match the number of interpolation conditions plus extra conditions.

(c) Consider the vector space of cubic splines, as in part (a), and let $d$ be the dimension of the vector space. Is there a basis $\varphi_1(x), \ldots, \varphi_d(x)$ with the following 'cardinality' property

$$\varphi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{for } i, j = 1, \ldots, n + 1$$

$$\varphi_j(x_i) = 0 \text{ for } j > n + 1 \text{ and } i = 1, \ldots, d.$$

If so, describe a method to construct $\{\varphi_i\}$; otherwise explain why it's not possible.

**5. Approximation** Let $w(x)$ be a weight function on the interval $[0, 1]$, and let $\phi_0(x), \phi_1(x), \ldots$ be a family of orthogonal polynomials with respect to the weight function $w(x)$ (with the usual assumption that $\phi_k(x)$ is a polynomial of degree $k$). Let $f(x) \in C[0, 1]$; you wish to find the polynomial $p(x)$ of degree $\leq n$ that minimizes

$$\int_0^1 x w(x)(f(x) - p(x))^2 \mathrm{d}x.$$

Describe an algorithm to compute the coefficients $c_k$ of the optimal polynomial with respect to the basis $\{\phi_k(x)\}_0^n$. The algorithm should require $\mathcal{O}(n)$ floating-point operations, not counting the cost to compute any norms or inner products of the basis functions and $f$.