Model-independent Superhedging under Portfolio Constraints

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Consider a market with a stock S.

The Superhedging Problem

Given a (path-dependent) payoff function Φ, what is the **minimal initial capital** needed to outperform the claim $\Phi(\{S_t\}_{0\le t\le T})$?

1. **Formulate the problem:** Take a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports the process S, and consider

$$D(\Phi) := \inf\{a \in \mathbb{R} : \exists \Delta \in \mathcal{H} \text{ s.t. } a + (\Delta \cdot S)_T \ge \Phi \ \mathbb{P}\text{-a.s.}\}.$$

   - $\mathcal{H} :=$ {admissible trading strategies}.
   - $(\Delta \cdot S)_T := \int_0^T \Delta_t dS_t$.

2. **Risk-neutral pricing:** Find probabilities $\mathbb{Q} \ll \mathbb{P}$ s.t. S is a $\mathbb{Q}$-martingale. Then,

$$D(\Phi) = \sup_{\mathbb{Q}\in\mathcal{Q}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}[\Phi], \tag{1}$$

where $\mathcal{Q}(\mathbb{P}) := \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P} \text{ is a martingale measure}\}$.

Some critiques on $\mathcal{Q}(\mathbb{P})$

Dupire (1994): liquidly traded options (e.g. vanilla calls) should be viewed as primary assets, with **prices given exogenously**.

- Let $C(t, K)$ denote the market price of a **vanilla call** with maturity $t > 0$ and strike $K > 0$.
- For any $t > 0$ and any pricing measure $\mathbb{Q}$,

$$\int_{\mathbb{R}_+} (S_t - K)^+ d\mathbb{Q} = C(t, K), \quad \forall K \geq 0.$$

- This already specifies the distribution of S_t under $\mathbb{Q}$.

$$\mu_t(K) = 1 - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(C(t, K) - C(t, K + \varepsilon) \right).$$

Conclusion: consider pricing measures $\mathbb{Q}$ under which

S_t admits the distribution μ_t for all $t \geq 0$.

Model-independent Superhedging

Difficult to find an appropriate physical measure $\mathbb{P}$ to start with.
$\Rightarrow$ Can we do superhedging **without** any a priori given $\mathbb{P}$?

Model-independent Superhedging

Can my terminal wealth $\geq$ a claim Φ,
no matter which probability $\mathbb{P}$ eventually materializes?

- Pioneering work: Hobson (1998).
- Extensions: Brown, Hobson & Rogers (2001), Bertsimas & Popescu (2002), Hobson, Laurence & Wang (2005), Cox & Obłój (2011), Dolinsky & Soner (2013),...

Most of the papers above

- focus on **specific** contingent claims
  (e.g. barrier, lookback, basket, double no-touch options).
- consider market prices of vanilla calls with **maturities at** T.

Our Framework

We start with the set-up in **Beiglböck, Henry-Labordère & Penkner (2013)**.

Consider a discrete-time market with finite horizon $T \in \mathbb{N}$.

- $\Omega := \mathbb{R}_+^T$.
- The stock S is taken as the coordinate mapping process, i.e.

$$S_t(x) = x_t \text{ for all } x = (x_1, \cdots, x_T) \in \mathbb{R}_+^T.$$

- $\mathbb{F} = \{\mathcal{F}_t\}_{t=1}^T$ is the natural filtration generated by S.
- Market prices $C(t, K)$ of **vanilla calls** for all maturities $t = 1, \cdots, T$ and strikes $K \geq 0$.
  - $\Rightarrow$ for each t, μ_t (the distribution of S_t) is specified.

We consider

$$\Pi := \{\mathbb{Q} \text{ probability on } \mathbb{R}_+^T : \mathbb{Q} \text{ admits marginals } \mu_1, \cdots, \mu_T\}.$$

This collection is **non-empty** and **weakly compact** (Villani (2009), Kellerer (1984)).

Note that $S_1, S_2, \cdots, S_T$ are $\mathbb{Q}$-integrable, for any $\mathbb{Q} \in \Pi$.

$$\mathbb{E}^{\mathbb{Q}}[S_t] = \int x \, d\mu_t(x) = C(t, 0).$$

Trading strategies in "stock":

- $\Delta = \{\Delta_t\}_{t=0}^{T-1}$ is a trading strategy if

$$\Delta_t(x_1, \cdots, x_t) \text{ is Borel measurable, for all } t.$$

- The stochastic integral is defined as

$$(\Delta \cdot x)_t := \sum_{i=0}^{t-1} \Delta_i(x_1, \cdots, x_i)(x_{i+1} - x_i), \quad \text{for } t = 1, \cdots, T.$$

- We denote by $\mathcal{H}$ the set of all trading strategies.

Static positions in "cash and vanilla calls":

- $u = \{u_t\}_{t=1}^{T}$ is a static position if each u_t is of the form

$$\varphi(x) = a + \sum_{i=1}^{n} b_i(x - K_i)^+,$$

for some $a \in \mathbb{R}$, $n \in \mathbb{N}$, $b_i \in \mathbb{R}$ and $K_i \geq 0$.

- We denote by $\mathcal{U}$ the set of all static positions.

Given a payoff Φ, want to find $\Delta \in \mathcal{H}$ and $u \in \mathcal{U}$ such that

$$\sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot x)_T \geq \Phi(x), \quad \forall x = (x_1, \cdots, x_T) \in \mathbb{R}_+^T. \quad (2)$$

Model-independent superhedging price

$$D(\Phi) := \inf \left\{ \sum_{t=1}^{T} \int_{\mathbb{R}_+} u_t d\mu_t : u \in \mathcal{U} \text{ and } \exists \Delta \in \mathcal{H} \text{ s.t. (2) holds} \right\}.$$

To get superhedging duality, the **pricing measures** should be??

$$\text{“}\mathbb{P}\text{”} \implies \mathcal{Q}(\mathbb{P}) = \{\mathbb{Q} \text{ mart. measure} : \mathbb{Q} \ll \mathbb{P}\}$$

$$\text{“} \quad \text{”} \implies \mathcal{M} := \{\mathbb{Q} \text{ mart. measure} : \mathbb{Q} \text{ admits marginals } \mu_t, \forall t\}.$$

Beiglböck, Henry-Labordère & Penkner (2013) use theory of "optimal transport" to prove the **superhedging duality**

$$D(\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi], \quad \mathcal{M} = \{\mathbb{Q} \in \Pi : \mathbb{Q} \text{ is a mart. measure}\}.$$

Acciaio, Beiglböck, Penkner & Schachermayer (2013) prove a **model-independent version of FTAP**

There is no model-independent arbitrage $\iff$ $\mathcal{M} \neq \emptyset$

MODEL-INDEPENDENT ARBITRAGE

There is model-independent arbitrage if $\exists \Delta \in \mathcal{H}$ and $u \in \mathcal{U}$ with $\sum_{t=1}^{T} \int u_t d\mu_t = 0$ s.t.

$$\sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot x)_T > 0, \quad \forall x \in \mathbb{R}_+^T.$$

What if: trading strategies are subject to constraints?

Semi-static superhedging under portfolio constraints

$$D(\Phi) := \inf \left\{ \sum_{t=1}^{T} \int_{\mathbb{R}_+} u_t d\mu_t : u \in \mathcal{U} \text{ and } \exists\ \Delta \in \mathcal{S} \text{ s.t. (2) holds} \right\},$$

where $\mathcal{S}$ is a subset of $\mathcal{H}$.

Our goals:

- model-independent duality for superhedging with $\Delta \in \mathcal{S}$.
- model-independent FTAP with $\Delta \in \mathcal{S}$.
- Examples and extensions

The Constraint Set $\mathcal{S}$

Definition

$\mathcal{S}$ is a collection of trading strategies such that

(i) $0 \in \mathcal{S}$.

(ii) [adapted convexity] For any $\Delta, \Delta' \in \mathcal{S}$ and any adapted process h with $h_t \in [0, 1]$ for all $t = 0, \cdots, T-1$,

$$h_t \Delta_t + (1 - h_t)\Delta'_t \in \mathcal{S}.$$

(iii) ... (TBA)

- (ii) is borrowed from Föllmer & Schied (2004).
- This already covers **convex Delta constraints** (and more...)

Upper Variation Process $A^{\mathbb{Q}}$

Introduced in Föllmer & Kramkov (1997), **upper variation process** was used to get some **supermartingale** property under portfolio constraints.

(Discrete) Upper Variation Process

For $\mathbb{Q} \in \Pi$, the upper variation process $A^{\mathbb{Q}}$ is defined by

$$\begin{aligned} A_0^{\mathbb{Q}} &:= 0, \\ A_{t+1}^{\mathbb{Q}} - A_t^{\mathbb{Q}} &:= \operatorname*{ess\,sup}_{\Delta \in \mathcal{S}}{}^{\mathbb{Q}} \left\{ \Delta_t (\mathbb{E}^{\mathbb{Q}}[S_{t+1} \mid \mathcal{F}_t] - S_t) \right\}, \quad t > 0 \\ &= \operatorname*{ess\,sup}_{\Delta \in \mathcal{S}^\infty}{}^{\mathbb{Q}} \left\{ \Delta_t (\mathbb{E}^{\mathbb{Q}}[S_{t+1} \mid \mathcal{F}_t] - S_t) \right\}, \quad t > 0 \end{aligned}$$

where

$$\mathcal{S}^\infty := \{\Delta \in \mathcal{S} : \Delta_t \text{ is bounded}, \ \forall t\}.$$

Basic Property of $A^{\mathbb{Q}}$

Lemma 1

For any $\mathbb{Q} \in \Pi$,

$$\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T]. \tag{3}$$

Idea: By the definition of $A_T^{\mathbb{Q}}$,

$$\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{Q}} \left[\operatorname*{ess\,sup}_{\Delta \in \mathcal{S}^\infty}{}^{\mathbb{Q}}\, \mathbb{E}^{\mathbb{Q}}[\Delta_t(S_{t+1} - S_t) \mid \mathcal{F}_t] \right].$$

For each $t > 0$, thanks to adapted convexity, the collection $\{\mathbb{E}^{\mathbb{Q}}[\Delta_t(S_{t+1} - S_t) \mid \mathcal{F}_t] : \Delta \in \mathcal{S}^\infty\}$ is directed upward. Thus,

$$\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \sum_{t=1}^{T} \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[\Delta_{t-1}(S_t - S_{t-1})] = \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T],$$

where the last equality follows from adapted convexity.

DEFINITION

Let $\mathcal{Q}_{\mathcal{S}}$ be the collection of $\mathbb{Q} \in \Pi$ such that

$$\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T] < \infty.$$

LEMMA 2

Given $\Delta \in \mathcal{S}$, $(\Delta \cdot S)_t - A_t^{\mathbb{Q}}$ is a local supermartingale, $\forall \mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$.

Idea: By the definition of $A_T^{\mathbb{Q}}$,

$$\mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_{t+1} - (\Delta \cdot S)_t \mid \mathcal{F}_t] = \Delta_t \cdot (\mathbb{E}^{\mathbb{Q}}[S_{t+1} \mid \mathcal{F}_t] - S_t) \leq A_{t+1}^{\mathbb{Q}} - A_t^{\mathbb{Q}},$$

$$\text{i.e. } \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_{t+1} - A_{t+1}^{\mathbb{Q}} \mid \mathcal{F}_t] \leq (\Delta \cdot S)_t - A_t^{\mathbb{Q}}. \quad (4)$$

But since $(\Delta \cdot S)_t$ may not lie in $L^1(\mathbb{Q}) \Rightarrow$ **local** supermartingality.

Supermartingale Property from $A^{\mathbb{Q}}$

Lemma 3

Fix $\Delta \in \mathcal{S}$ and $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$. If $(\Delta \cdot S)_T \geq \varphi$ with φ $\mathbb{Q}$-integrable, then

$$(\Delta \cdot S)_t - A_t^{\mathbb{Q}} \geq \mathbb{E}^{\mathbb{Q}}[\varphi - A_T^{\mathbb{Q}} \mid \mathcal{F}_t] \quad \mathbb{Q}\text{-a.s.}, \quad \forall t. \tag{5}$$

This implies $(\Delta \cdot S)_t - A_t^{\mathbb{Q}}$ is a true $\mathbb{Q}$-supermartingale.

Idea: Prove this by induction. At time T, (5) trivially holds true as

$$(\Delta \cdot S)_T - A_T^{\mathbb{Q}} \geq \varphi - A_T^{\mathbb{Q}}. \tag{6}$$

Since $(\Delta \cdot S)_t - A_t^{\mathbb{Q}}$ is a local supermartingale, can use (6) to prove

$$(\Delta \cdot S)_{T-1} - A_{T-1}^{\mathbb{Q}} \geq \mathbb{E}^{\mathbb{Q}}[\varphi - A_T^{\mathbb{Q}} \mid \mathcal{F}_{T-1}].$$

Weak duality $P(\Phi) \leq D(\Phi)$

Proposition 1

Suppose $\Phi : \mathbb{R}_+^T \mapsto \mathbb{R}$ is measurable and $\exists K > 0$ s.t.

$$|\Phi(x_1, \cdots, x_T)| \leq K(1 + x_1 + \cdots + x_T), \ \forall x \in \mathbb{R}_+^T. \quad (7)$$

Then,

$$P(\Phi) := \sup_{\mathbb{Q} \in \mathcal{Q}_\mathcal{S}} \mathbb{E}^\mathbb{Q}[\Phi - A_T^\mathbb{Q}] \leq D(\Phi). \quad (8)$$

Idea: Take $u \in \mathcal{U}$ and $\Delta \in \mathcal{S}$ s.t. $\sum_{t=1}^T u_t(x_t) + (\Delta \cdot x)_T \geq \Phi$. For any $\mathbb{Q} \in \mathcal{Q}_\mathcal{S}$, note that

$$(\Delta \cdot S)_T \geq \varphi(x) := \Phi(x) - \sum_{t=1,\cdots,T} u_t(x_t),$$

and φ is $\mathbb{Q}$-integrable thanks to (7). Thus, Lemma 3 gives

$$\mathbb{E}^\mathbb{Q}[\Phi - A_T^\mathbb{Q}] \leq \mathbb{E}^\mathbb{Q}\left[\sum_{t=1}^T u_t(S_t) + (\Delta \cdot S)_T - A_T^\mathbb{Q}\right] \leq \sum_{t=1}^T \int_{\mathbb{R}_+} u_t d\mu_t.$$

Proving $P(\Phi) \geq D(\Phi)$

$$D(\Phi) \leq \inf \left\{ \sum_{t=1}^{T} \int_{\mathbb{R}_+} u_t d\mu_t : \exists \Delta \in \mathcal{S}_c^\infty \text{ s.t. } \sum_t u_t + (\Delta \cdot x)_T \geq \Phi(x) \right\}$$

$$= \inf_{\Delta \in \mathcal{S}_c^\infty} \inf \left\{ \sum_{t=1}^{T} \int u_t d\mu_t : \sum_{t=1}^{T} u_t(x_t) \geq \Phi(x) - (\Delta \cdot x)_T \right\}$$

Monge-Kantorovich Duality

Let $\varphi : \mathbb{R}_+^T \mapsto \mathbb{R}$ be upper semi-continuous and $\exists K > 0$ such that

$$|\varphi(x_1, \cdots, x_T)| \leq K(1 + x_1 + \cdots + x_T), \ \forall x \in \mathbb{R}_+^T.$$

Then,

$$\sup_{\mathbb{Q} \in \Pi} \mathbb{E}^{\mathbb{Q}}[\varphi] = \inf \left\{ \sum_{t=1}^{T} \int u_t d\mu_t : u_1(x_1) + \cdots + u_T(x_T) \geq \varphi(x) \right\}.$$

Proving $P(\Phi) \geq D(\Phi)$

$$\implies \quad D(\Phi) \leq \inf_{\Delta \in \mathcal{S}_c^\infty} \sup_{\mathbb{Q} \in \Pi} \mathbb{E}^{\mathbb{Q}}[\Phi(x) - (\Delta \cdot x)_T]$$

Minimax Theorem (Sion)

Let X be a compact convex subset of a vector space, Y be a convex subset of a vector space, and $f : X \times Y \mapsto \mathbb{R}$ satisfy

(i) Given $x \in X$, $y \mapsto f(x, y)$ is convex on Y.

(ii) Given $y \in Y$, $x \mapsto f(x, y)$ is upper semi-continuous and concave on X.

Then,

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

Taking $X = \Pi$, $Y = \mathcal{S}_c^\infty$ and $f(\mathbb{Q}, \Delta) = \mathbb{E}^{\mathbb{Q}}[\Phi(x) - (\Delta \cdot x)_T]$,

$$D(\Phi) \leq \sup_{\mathbb{Q} \in \Pi} \inf_{\Delta \in \mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}}[\Phi(x) - (\Delta \cdot x)_T]$$

(assuming Φ is u.s.c.)

PROVING $P(\Phi) \geq D(\Phi)$

$$\begin{aligned}
\Longrightarrow \quad D(\Phi) &\leq \sup_{\mathbb{Q}\in\Pi} \left\{ \mathbb{E}^{\mathbb{Q}}[\Phi] - \sup_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta\cdot x)_T] \right\} \\
&= \sup_{\mathbb{Q}\in\Pi} \left\{ \mathbb{E}^{\mathbb{Q}}[\Phi] - \sup_{\Delta\in\mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta\cdot S)_T] \right\} \\
&= \sup_{\mathbb{Q}\in\Pi} \left\{ \mathbb{E}^{\mathbb{Q}}[\Phi] - \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] \right\} \\
&= \sup_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S}}} \left\{ \mathbb{E}^{\mathbb{Q}}[\Phi] - \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] \right\} = P(\Phi),
\end{aligned}$$

Q: how can we guarantee that

$$\sup_{\Delta\in\mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta\cdot x)_T] = \sup_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta\cdot x)_T] \ ?$$

The Constraint Set $\mathcal{S}$

Definition 1

$\mathcal{S}$ is a collection of trading strategies such that

(i) $0 \in \mathcal{S}$.

(ii) [adapted convexity] For any $\Delta, \Delta' \in \mathcal{S}$ and any adapted process h with $h_t \in [0,1]$ for all $t = 0, \cdots, T-1$,

$$h_t \Delta_t + (1-h_t)\Delta'_t \in \mathcal{S}.$$

(iii) [continuous approximation] Given $\Delta \in \mathcal{S}^\infty$, $\mathbb{Q} \in \Pi$, and $\varepsilon > 0$, $\exists$ closed $D_\varepsilon \subseteq \mathbb{R}_+^T$ and $\Delta^\varepsilon \in \mathcal{S}_c^\infty$ s.t.

$$\mathbb{Q}(D_\varepsilon) > 1 - \varepsilon \quad \text{and} \quad \Delta_t = \Delta_t^\varepsilon \text{ on } D_\varepsilon \ \forall t.$$

Lemma 4

Under Definition 1 (iii),

$$\sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot x)_T] = \sup_{\Delta \in \mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot x)_T].$$

The Constraint Set $\mathcal{S}$

Definition 1 (iii) is not very restrictive, as it covers

- **Deterministic convex constraints:**
  For each t, let $K_t \subseteq \mathbb{R}$ be a closed convex set. Then

$$\mathcal{S} := \{\Delta \in \mathcal{H} : \text{for each } t,\ \Delta_t(x) \in K_t \ \ \forall x \in \mathbb{R}^t_+\}$$

  satisfies Definition 1 (iii), thanks to **Lusin's theorem** and **continuous extension** theorem.

- **Adapted convex constraints:**
  Let $\{K_t\}_{t=0}^T$ be an adapted set-valued process such that for each t, $K_t(x) = [m_t(x), M_t(x)]$ $\forall x \in \mathbb{R}^t_+$. Then

$$\mathcal{S} := \{\Delta \in \mathcal{H} : \text{for each } t,\ \Delta_t(x) \in K_t(x) \ \forall x \in \mathbb{R}^t_+\}.$$

  satisfies Definition 1 (iii), if m_t and M_t are continuous [thanks to **continuous selection** theory in Michael (1956) and Brown (1989)].

THE DUALITY

Suppose $\Phi : \mathbb{R}_+^T \mapsto \mathbb{R}$ is upper semi-continuous and $\exists K > 0$ s.t.

$$|\Phi(x_1, \cdots, x_T)| \leq K(1 + x_1 + \cdots + x_T), \quad \forall x \in \mathbb{R}_+^T. \tag{9}$$

Then

$$P(\Phi) := \sup_{\mathbb{Q} \in \mathcal{Q}_\mathcal{S}} \mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}] = D(\Phi).$$

If $\mathcal{Q}_\mathcal{S} \neq \emptyset$, then $\exists\ \mathbb{Q}^* \in \mathcal{Q}_\mathcal{S}$ s.t. $P(\Phi) = \mathbb{E}^{\mathbb{Q}^*}[\Phi - A_T^{\mathbb{Q}^*}]$.

Consider

$$\mathcal{X} := \{\Phi : \mathbb{R}_+^T \mapsto \mathbb{R} : \Phi \text{ satisfies (9) } \textbf{[linear growth]}\}.$$

CONVEX RISK MEASURE

$\rho : \mathcal{X} \mapsto \mathbb{R}$ is called a **convex risk measure** if for all $\Phi, \Phi' \in \mathcal{X}$,

- [Monotonicity] If $\Phi \leq \Phi'$, then $\rho(\Phi) \geq \rho(\Phi')$.
- [Translation Invariance] If $m \in \mathbb{R}$, then $\rho(\Phi + m) = \rho(\Phi) - m$.
- [Convexity] If $0 \leq \lambda \leq 1$, then

$$\rho(\lambda\Phi + (1-\lambda)\Phi') \leq \lambda\rho(\Phi) + (1-\lambda)\rho(\Phi').$$

Let $\rho_{\mathcal{S}} : \mathcal{X} \mapsto \mathbb{R}$ be defined by

$$\rho_{\mathcal{S}}(\Phi) := D(-\Phi).$$

PROPOSITION

Suppose $\mathcal{Q}_{\mathcal{S}} \neq \emptyset$. Then, $\rho_{\mathcal{S}} := D(-\Phi)$ is a convex risk measure, and it admits the dual formulation

$$\rho_{\mathcal{S}}(\Phi) = \sup_{\mathbb{Q}\in\Pi} \left(\mathbb{E}^{\mathbb{Q}}[-\Phi] - \alpha^*(\mathbb{Q})\right), \tag{10}$$

where the **penalty function** α^* *is given by*

$$\alpha^*(\mathbb{Q}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] & \text{if } \mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}, \\ \infty, & \text{otherwise.} \end{cases}$$

This generalizes Föllmer & Schied (2002) to a model-independent framework. Moreover,

- we cover **unbounded** financial positions (with linear growth).
- our assumption "$\mathcal{Q}_{\mathcal{S}} \neq \emptyset$" is weaker than "no arbitrage".

Model-independent Arbitrage

Arbitrage under Constraints

There is **model-independent arbitrage** under constraint $\mathcal{S}$, if $\exists\ u \in \mathcal{U}$ with $\sum_{t=1}^{T} \int u_t d\mu_t = 0$ and $\Delta \in \mathcal{S}$ s.t.

$$\sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot x)_T > 0, \quad \forall\ x \in \mathbb{R}_+^T.$$

Properties of $\mathcal{P}_{\mathcal{S}}$

Consider the set of probability measures

$$\mathcal{P}_{\mathcal{S}} := \{\mathbb{Q} \in \Pi : (\Delta \cdot S)_t \text{ is a local } \mathbb{Q}\text{-supermartingale}, \ \forall \Delta \in \mathcal{S}\}.$$

Lemma 5

Fix $\mathbb{Q} \in \Pi$. Then, $\mathbb{Q} \in \mathcal{P}_{\mathcal{S}} \iff A_T^{\mathbb{Q}} = 0$ $\mathbb{Q}$-a.s.

Idea: $(\Leftarrow)$ Obvious, as $(\Delta \cdot S)_t - A_t^{\mathbb{Q}}$ is a local supermartingale.

$(\Rightarrow)$ Given $\Delta \in \mathcal{S}^\infty$, $(\Delta \cdot S)_t$ is a local $\mathbb{Q}$-supermartingale. Boundedness of Δ implies $(\Delta \cdot S)^-$ is $\mathbb{Q}$-integrable, and thus $(\Delta \cdot S)_t$ is a true supermartingale. Then,

$$\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T] = 0.$$

Consequence:

- $\mathcal{P}_{\mathcal{S}} \subseteq \mathcal{Q}_{\mathcal{S}}$.
- If $\mathcal{P}_{\mathcal{S}} = \emptyset$, $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] > 0$ for all $\mathbb{Q} \in \Pi \Rightarrow \inf_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] > 0$?

LEMMA 6

$$\text{If } \mathcal{P}_{\mathcal{S}} = \emptyset, \text{ then } \inf_{\mathbb{Q}\in\Pi} \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] > 0.$$

Idea: Suppose $\inf_{\mathbb{Q}\in\Pi} \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = 0$. Then, $\forall\ \varepsilon > 0$, $\exists\ \mathbb{Q}_\varepsilon \in \Pi$ s.t. $0 \le \mathbb{E}^{\mathbb{Q}_\varepsilon}[A_T^{\mathbb{Q}_\varepsilon}] < \varepsilon$. Since $\mathbb{Q}_\varepsilon$ converges weakly to some $\mathbb{Q}^* \in \Pi$ (recall that Π is weakly compact),

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \mathbb{E}^{\mathbb{Q}_\varepsilon}[A_T^{\mathbb{Q}_\varepsilon}] = \lim_{\varepsilon\to 0} \sup_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}_\varepsilon}[(\Delta\cdot S)_T] \\ &\ge \sup_{\Delta\in\mathcal{S}_c^\infty} \lim_{\varepsilon\to 0} \mathbb{E}^{\mathbb{Q}_\varepsilon}[(\Delta\cdot S)_T] = \sup_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}^*}[(\Delta\cdot S)_T] = \mathbb{E}^{\mathbb{Q}^*}[A_T^{\mathbb{Q}^*}]. \end{aligned}$$

Thus, $A_T^{\mathbb{Q}^*} = 0$ $\mathbb{Q}^*$-a.s. By Lemma 5, $\mathbb{Q}^* \in \mathcal{P}_{\mathcal{S}}$, a contradiction.

Note: for "$=$", since $(\Delta\cdot S)_T$ may not be bounded, need additional estimates from Villani (2009).

FTAP under Constraints

The following are equivalent.

(i) There is no model-independent arbitrage under constraint $\mathcal{S}$.

(ii) $\mathcal{P}_{\mathcal{S}} \neq \emptyset$.

Idea: [(ii) $\Rightarrow$ (i)] Suppose there is model-independent arbitrage, i.e. $\exists\ u \in \mathcal{U}$ with $\sum_t \int u_t d\mu_t = 0$ and $\Delta \in \mathcal{S}$ s.t.

$$\sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot S)_T > 0 \quad \forall x \in \mathbb{R}_+^T.$$

$$\Longrightarrow \quad \sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot S)_T - A_T^{\mathbb{Q}} > -A_T^{\mathbb{Q}} \quad \mathbb{Q}\text{-a.s.}, \quad \forall \mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}.$$

$$\Longrightarrow \quad 0 \geq \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T - A_T^{\mathbb{Q}}] > -\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}], \quad \forall \mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}.$$

Hence, $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] > 0\ \forall \mathbb{Q} \in \mathcal{Q}_{\mathcal{S}} \Rightarrow \mathbb{Q} \notin \mathcal{P}_{\mathcal{S}}\ \forall \mathbb{Q} \in \mathcal{Q}_{\mathcal{S}} \Rightarrow \mathcal{P}_{\mathcal{S}} = \emptyset$.

[(i) ⇒ (ii)] Suppose $\mathcal{P}_{\mathcal{S}} = \emptyset$. By Lemma 6,

$$\delta := \inf_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S}}} \mathbb{E}^{\mathbb{Q}}[A^{\mathbb{Q}}_T] \geq \inf_{\mathbb{Q}\in\Pi} \mathbb{E}^{\mathbb{Q}}[A^{\mathbb{Q}}_T] > 0.$$

Taking $\Phi \equiv 0$ in Superhedging Duality,

$$D(0) = \sup_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S}}} \mathbb{E}^{\mathbb{Q}}[-A^{\mathbb{Q}}_T] = -\delta.$$

This implies: can superhedge $\Phi \equiv 0$ with initial wealth $-\delta/2$, i.e. $\exists\ u \in \mathcal{U}$ with $\sum_{t=1}^{T} \int_{\mathbb{R}_+} u_t d\mu_t = -\delta/2$ and $\Delta \in \mathcal{S}$ s.t.

$$\sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot S)_T \geq 0 \quad \forall x \in \mathbb{R}^T_+.$$

$$\implies \quad \sum_{t=1}^{T} \left(u_t(x_t) - \int_{\mathbb{R}_+} u_t d\mu_t \right) + (\Delta \cdot S)_T \geq 0 + \frac{\delta}{2} > 0 \quad \forall x \in \mathbb{R}^T_+.$$

This is already arbitrage with $u'_t(z) := u_t(z) - \int_{\mathbb{R}_+} u_t d\mu_t$ and Δ.

Characterizing $\mathcal{Q}_{\mathcal{S}} \neq \emptyset$

Superhedging and **risk-measuring** are meaningful as long as

$$\mathcal{Q}_{\mathcal{S}} \neq \emptyset,$$

which is *weaker than* the no-arbitrage condition $\mathcal{P}_{\mathcal{S}} \neq \emptyset$.

Model-independent unbounded profit

There is **model-independent unbounded profit** under constraint $\mathcal{S}$, if $\forall\ a \in \mathbb{R}_+$, $\exists\ u \in \mathcal{U}$ with $\sum_{t=1}^{T} \int u_t d\mu_t = 0$ and $\Delta \in \mathcal{S}$ s.t.

$$\sum_{t=1}^{T} u_t(x_t) + (\Delta \cdot x)_T > a, \quad \forall x \in \mathbb{R}_+^T. \tag{11}$$

FTAP for Unbounded Profit

The following are equivalent.

(I) There is no model-independent unbounded profit under $\mathcal{S}$.

(II) $\mathcal{Q}_{\mathcal{S}} \neq \emptyset$.

If **no constraint**, i.e. $\mathcal{S} = \mathcal{H}$, can show that

$$\mathcal{M} = \mathcal{P}_{\mathcal{S}} = \mathcal{Q}_{\mathcal{S}}.$$

SUPERHEDEING DUALITY

Suppose $\mathcal{S} = \mathcal{H}$. Let Φ be u.s.c. and has linear growth. Then

$$D(\Phi) = \sup_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}} \mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}] = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi].$$

This recovers the duality in Beiglböck et al. (2013).

FTAP

The following are equivalent:

(I) There is no model-independent arbitrage for $\Delta \in \mathcal{H}$.

(II) $\mathcal{M} \neq \emptyset$.

This recovers FTAP in Acciaio et al. (2013).

In Acciaio et al. (2013),

- tradable options can be very general.
- use functional analysis (Stone-Cech compactification)
- FTAP $\Rightarrow$ Superhedging duality.

In our paper,

- tradable options: vanilla calls with all maturities and strikes.
- use duality from optimal transport, weak compactness.
- Superhedging duality $\Rightarrow$ FTAP.

Shortselling constraint: Given $c \in \mathbb{R}_+$, consider

$$\mathcal{S} := \{\Delta \in \mathcal{H} : \Delta_t \geq -c, \ \forall t\}.$$

We have $\mathcal{P}_{\mathcal{S}} = \mathcal{Q}_{\mathcal{S}} = \{\mathbb{Q} \in \Pi : S \text{ is a } \mathbb{Q}\text{-supermartingale}\}$

no arbitrage $\iff$ no unbounded profit.

Relative-drawdown constraint: Consider the running maximum

$$x_t^* := \max\{x_0, x_1, \cdots, x_t\}.$$

For any continuous functions a and b, introduce

$$\mathcal{S} := \{\Delta \in \mathcal{H} : a(S_t/S_t^*) \leq \Delta_t \leq b(S_t/S_t^*), \ \forall t\}.$$

We have $\mathcal{Q}_{\mathcal{S}} = \Pi$. $\implies$ no unbounded profit under $\mathcal{S}$

Gamma Constraint

Given $\Gamma > 0$, consider

$$\mathcal{S}_\Gamma := \{\Delta \in \mathcal{H} : |\Delta_t - \Delta_{t-1}| \leq \Gamma,\ \forall t\}, \quad \text{where } \Delta_{-1} \equiv 0.$$

Note:

- $\mathcal{S}_\Gamma$ does NOT satisfy [adapted convexity].
  - $\Delta \equiv 0$, $\Delta' \equiv 2\Gamma \in \mathcal{S}_\Gamma$, but $\tilde{\Delta}_s := 0\ 1_{\{s \leq 2\}} + 2\Gamma\ 1_{\{s>2\}} \notin \mathcal{S}_\Gamma$.
- Every $\Delta \in \mathcal{S}_\Gamma$ is **bounded**.

Using this **boundedness**, can show that superhedging duality and FTAP still hold true.

Proposition

$$\mathcal{Q}_{\mathcal{S}_\Gamma} = \Pi \ \neq \emptyset \text{ and } \mathcal{P}_{\mathcal{S}_\Gamma} = \mathcal{M}.$$

Future Work

- Can we take into account other type of frictions?
  - transaction costs; Dolinsky & Soner (2013).
  - find a **unified approach** to deal with different kinds of frictions.
- Can we drop the semi-continuity condition?
  - Do not need this in classical case,
    nor in the **model uncertainty** framework by Bouchard & Nutz (2013).
  - quantile hedging, hedging under controlled loss, ...

THANK YOU very much for your attention!

Q & A

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