Remember to write your name! You are allowed to use a calculator. You are not allowed to use the textbook, your notes, the internet, or your neighbor. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

**Name:**

1. (30 points) If the statement is always true mark “TRUE”; if it is possible for the statement to be false then mark “FALSE.” If the statement seems neither true nor false but rather incoherent, raise your hand. No justification is necessary. **Students in 4720 can pick 5 out of 6 questions to answer. Students in 5720 must answer all.**

   (a) If a matrix $A$ is normal then the eigenvalues of the perturbed matrix $A + E$ are all within a distance $\|E\|_3$ of the eigenvalues of $A$.

   **False** This is the Bauer-Fike theorem. You can use any $p$-norm. But in the 3-norm the condition of the eigenvector basis is not necessarily 1, as it would be in the 2-norm.

   (b) Let $\lambda$ be a simple eigenvalue of $A$ with left and right eigenvectors $y$ and $x$, each of which are 2-norm unit vectors, and let $A + E$ be the perturbed matrix where $\|E\|_2 = \epsilon$. True or False: The perturbed matrix will have an eigenvalue $\mu$ within a distance of approximately $\epsilon/|y^*x|$ from $\lambda$, for small-enough $\epsilon$.

   **True**

   (c) Let $A$ be a diagonalizable matrix with approximate eigenvalue/eigenvector pair $(\mu, x)$. True or false: $(\mu, x)$ is an exact eigenvalue/eigenvector pair for a perturbed matrix $A + E$ where $\|E\|_2 \leq \|Ax - \mu x\|_2$.

   **False** This would be true for *normal* matrices, but the correct statement for non-normal matrices includes the condition number of the eigenvector basis.

   (d) Suppose that $A$ is $n \times n$ with LU factorization $PA = LU$. True or false: The matrix $UP^T L$ has the same eigenvalues as $A$.

   **True** The matrices are similar:

   $$UP^T L = L^{-1} PAP^T L.$$  

   $L$ is always invertible (even if $A$ is not) because it is lower triangular with ones on the diagonal.

   (e) Let $\lambda$ and $x$ be an eigenvalue/eigenvector pair for $A$. True or false: The matrix $A - \lambda xx^*/\|x\|_2^2$ has eigenvector $x$ with eigenvalue 0.

   **True**

   (f) Let $S$ be a nontrivial subspace that is invariant under a square matrix $A$. True or False: There is an eigenvector of $A$ in $S$.

   **True**
2. (20 points) Suppose that you are given one eigenvalue/eigenvector pair of an $n \times n$ matrix $A$. Explain how you can reduce the problem of finding the remaining eigenvalues of $A$ to finding the eigenvalues of an $n-1 \times n-1$ matrix. Show explicitly how to construct the $n-1 \times n-1$ matrix. Hint: Start by constructing an invertible matrix $X$ whose first column is the eigenvector.

Let $X$ be an invertible matrix whose first column is the eigenvector. Then

$$X^{-1}AX = \begin{bmatrix} \lambda & * \\ 0 & B \end{bmatrix}.$$ 

$A$ is similar to the RHS, which is block-upper triangular. The eigenvalues of $A$ are therefore $\lambda$ together with the eigenvalues of $B$, which is $n-1 \times n-1$. Kudos if you used a unitary similarity transform rather than just an invertible $X$.

3. Computing the SVD of a real $m \times n$ matrix $A$ requires computing the eigenvalues and eigenvectors of $A^TA$ and $AA^T$.

(a) Let $P$ and $Q$ be real orthogonal matrices of size $m \times m$ and $n \times n$ respectively, and let $B = PAQ$. Show that the singular values of $B$ are the same as the singular values of $A$.

The singular values of $A$ are the square roots of the eigenvalues of $A^TA$, and similarly for the singular values of $B$. Note that

$$B^TB = Q^TA^TAQ$$ 

so $B^TB$ is (orthogonally-)similar to $A^TA$, and they therefore have the same eigenvalues.

(b) Let $v$ be an eigenvector of $B^TB$. How is it related to the corresponding eigenvector of $A^TA$?

The above analysis shows that if $v$ is an eigenvector of $B^TB$, then $Qv$ is an eigenvector of $A^TA$.

(c) It is possible to choose $P$ and $Q$ such that $B$ is bi-diagonal (nonzeros immediately above the diagonal). Prove that $B^TB$ and $BB^T$ are tridiagonal (you may cite any relevant theorem from class).

The banded-matrix-multiplication theorem shows that multiplying a lower-bidiagonal and an upper-bidiagonal matrix yields a tridiagonal matrix.
4. (20 points)

- **5720 Only** Let \( A \) be an \( n \times n \) diagonalizable matrix with eigenvalues satisfying \( \lambda_1 = \ldots = \lambda_k \) with \( |\lambda_k| > |\lambda_{k+1}| \geq \ldots \geq |\lambda_n| \). Show that the vectors generated by the power method will converge to an eigenvector of \( A \) (under standard assumptions on the starting vector).

Let the eigenvectors of \( A \) be \( v_1, \ldots, v_n \), and the initial vector for the power method be \( x_0 = c_1 v_1 + \cdots + c_n v_n \). Assume that \( c_1, \ldots, c_k \) are not all zero. Then

\[
A^p x_0 = c_1 \lambda_1^p v_1 + \cdots + c_k \lambda_1^p v_k + c_{k+1} \lambda_{k+1}^p v_{k+1} + \cdots + c_n \lambda_n^p v_n
\]

\[
A^p x_0 = \lambda_1^p (c_1 v_1 + \cdots + c_k v_k) + c_{k+1} \lambda_{k+1}^p v_{k+1} + \cdots + c_n \lambda_n^p v_n
\]

If you normalize then the coefficients of \( v_{k+1}, \ldots, v_n \) will decay to 0 as \( p \to \infty \) so that \( A^p x_0 / \|A^p x_0\| \) will converge to

\[
\frac{c_1 v_1 + \cdots + c_k v_k}{\|c_1 v_1 + \cdots + c_k v_k\|}
\]

This vector is an eigenvector of \( A \) because

\[
A(c_1 v_1 + \cdots + c_k v_k) = \lambda_1 (c_1 v_1 + \cdots + c_k v_k).
\]

- **4720 Only** The basic shifted QR algorithm is

\[
A_{m-1} - \rho I = Q_m R_m, \quad A_m = R_m Q_m + \rho I, \quad A_0 = A.
\]

Show that \( A_m \) is orthogonally similar to \( A_{m-1} \) (you may assume everything is real).

\[
A_{m-1} = Q_m R_m + \rho I \Rightarrow Q_m^T A_{m-1} Q_m = R_m Q_m + \rho I = A_m
\]