Remember to write your name! You are allowed to use a calculator. You are not allowed to use the textbook, your notes, the internet, or your neighbor. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

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1. (30 points) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." If the statement seems neither true nor false but rather incoherent, raise your hand. No justification is necessary. **Students in 4720 can pick 5 out of 6 questions to answer. Students in 5720 must answer all.**

_____(a) If a matrix $\mathbf{A}$ is normal then the eigenvalues of the perturbed matrix $\mathbf{A} + \mathbf{E}$ are all within a distance $\|\mathbf{E}\|_3$ of the eigenvalues of $\mathbf{A}$.

False This is the Bauer-Fike theorem. You can use any p-norm. But in the 3-norm the condition of the eigenvector basis is not necessarily 1, as it would be in the 2-norm.

_____(b) Let λ be a simple eigenvalue of $\mathbf{A}$ with left and right eigenvectors $\boldsymbol{y}$ and $\boldsymbol{x}$, each of which are 2-norm unit vectors, and let $\mathbf{A} + \mathbf{E}$ be the perturbed matrix where $\|\mathbf{E}\|_2 = \epsilon$. True or False: The perturbed matrix will have an eigenvalue μ within a distance of approximately $\epsilon/|\boldsymbol{y}^*\boldsymbol{x}|$ from λ, for small-enough ϵ.

True

_____(c) Let $\mathbf{A}$ be a diagonalizable matrix with approximate eigenvalue/eigenvector pair $(\mu, \boldsymbol{x})$. True or false: $(\mu, \boldsymbol{x})$ is an exact eigenvalue/eigenvector pair for a perturbed matrix $\mathbf{A} + \mathbf{E}$ where $\|\mathbf{E}\|_2 \leq \|\mathbf{A}\boldsymbol{x} - \mu\boldsymbol{x}\|_2$.

False This would be true for *normal* matrices, but the correct statement for non-normal matrices includes the condition number of the eigenvector basis.

_____(d) Suppose that $\mathbf{A}$ is $n \times n$ with LU factorization $\mathbf{PA} = \mathbf{LU}$. True or false: The matrix $\mathbf{UP}^T\mathbf{L}$ has the same eigenvalues as $\mathbf{A}$.

True The matrices are similar:

$$\mathbf{UP}^T\mathbf{L} = \mathbf{L}^{-1}\mathbf{PAP}^T\mathbf{L}.$$

$\mathbf{L}$ is always invertible (even if $\mathbf{A}$ is not) because it is lower triangular with ones on the diagonal.

_____(e) Let λ and $\boldsymbol{x}$ be an eigenvalue/eigenvector pair for $\mathbf{A}$. True or false: The matrix $\mathbf{A} - \lambda\boldsymbol{x}\boldsymbol{x}^*/\|\boldsymbol{x}\|_2^2$ has eigenvector $\boldsymbol{x}$ with eigenvalue 0.

True

_____(f) Let $\mathcal{S}$ be a nontrivial subspace that is invariant under a square matrix $\mathbf{A}$. True or False: There is an eigenvector of $\mathbf{A}$ in $\mathcal{S}$.

True

2. (20 points) Suppose that you are given one eigenvalue/eigenvector pair of an $n \times n$ matrix $\mathbf{A}$. Explain how you can reduce the problem of finding the remaining eigenvalues of $\mathbf{A}$ to finding the eigenvalues of an $n-1 \times n-1$ matrix. Show explicitly how to construct the $n-1 \times n-1$ matrix. Hint: Start by constructing an invertible matrix $\mathbf{X}$ whose first column is the eigenvector.

   Let $\mathbf{X}$ be an invertible matrix whose first column is the eigenvector. Then

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \left[\begin{array}{c|c} \lambda & * \\ \hline \mathbf{0} & \mathbf{B} \end{array}\right].$$

   $\mathbf{A}$ is similar to the RHS, which is block-upper triangular. The eigenvalues of $\mathbf{A}$ are therefore λ together with the eigenvalues of $\mathbf{B}$, which is $n-1 \times n-1$. Kudos if you used a unitary similarity transform rather than just an invertible $\mathbf{X}$.

3. Computing the SVD of a real $m \times n$ matrix $\mathbf{A}$ requires computing the eigenvalues and eigenvectors of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$.

   (a) Let $\mathbf{P}$ and $\mathbf{Q}$ be real orthogonal matrices of size $m \times m$ and $n \times n$ respectively, and let $\mathbf{B} = \mathbf{PAQ}$. Show that the singular values of $\mathbf{B}$ are the same as the singular values of $\mathbf{A}$.

   The singular values of $\mathbf{A}$ are the square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$, and similarly for the singular values of $\mathbf{B}$. Note that

$$\mathbf{B}^T\mathbf{B} = \mathbf{Q}^T\mathbf{A}^T\mathbf{A}\mathbf{Q}$$

   so $\mathbf{B}^T\mathbf{B}$ is (orthogonally-)similar to $\mathbf{A}^T\mathbf{A}$, and they therefore have the same eigenvalues.

   (b) Let $\boldsymbol{v}$ be an eigenvector of $\mathbf{B}^T\mathbf{B}$. How is it related to the corresponding eigenvector of $\mathbf{A}^T\mathbf{A}$?

   The above analysis shows that if $\boldsymbol{v}$ is an eigenvector of $\mathbf{B}^T\mathbf{B}$, then $\mathbf{Q}\boldsymbol{v}$ is an eigenvector of $\mathbf{A}^T\mathbf{A}$.

   (c) It is possible to choose $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{B}$ is bi-diagonal (nonzeros immediately above the diagonal). Prove that $\mathbf{B}^T\mathbf{B}$ and $\mathbf{B}\mathbf{B}^T$ are tridiagonal (you may cite any relevant theorem from class).

   The banded-matrix-multiplication theorem shows that multiplying a lower-bidiagonal and an upper-bidiagonal matrix yields a tridiagonal matrix.

4. (20 points)

- **5720 Only** Let $\mathbf{A}$ be an $n \times n$ diagonalizable matrix with eigenvalues satisfying $\lambda_1 = \ldots = \lambda_k$ with $|\lambda_k| > |\lambda_{k+1}| \geq \ldots \geq |\lambda_n|$. Show that the vectors generated by the power method will converge to an eigenvector of $\mathbf{A}$ (under standard assumptions on the starting vector).

  Let the eigenvectors of $\mathbf{A}$ be $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, and the initial vector for the power method be $\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$. Assume that $c_1, \ldots, c_k$ are not all zero. Then

$$\mathbf{A}^p\boldsymbol{x}_0 = c_1\lambda_1^p\boldsymbol{v}_1 + \cdots + c_k\lambda_1^p\boldsymbol{v}_k + c_{k+1}\lambda_{k+1}^p\boldsymbol{v}_{k+1} + \cdots + c_n\lambda_n^p\boldsymbol{v}_n$$

$$\mathbf{A}^p\boldsymbol{x}_0 = \lambda_1^p(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) + c_{k+1}\lambda_{k+1}^p\boldsymbol{v}_{k+1} + \cdots + c_n\lambda_n^p\boldsymbol{v}_n$$

  If you normalize then the coefficients of $\boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_n$ will decay to 0 as $p \to \infty$ so that $\mathbf{A}^p\boldsymbol{x}_0/\|\mathbf{A}^p\boldsymbol{x}_0\|$ will converge to

$$\frac{c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k}{\|c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k\|}.$$

  This vector is an eigenvector of $\mathbf{A}$ because

$$\mathbf{A}(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) = \lambda_1(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k).$$

- **4720 Only** The basic shifted QR algorithm is

$$\mathbf{A}_{m-1} - \rho\mathbf{I} = \mathbf{Q}_m\mathbf{R}_m, \quad \mathbf{A}_m = \mathbf{R}_m\mathbf{Q}_m + \rho\mathbf{I}, \quad \mathbf{A}_0 = \mathbf{A}.$$

  Show that $\mathbf{A}_m$ is orthogonally similar to $\mathbf{A}_{m-1}$ (you may assume everything is real).

$$\mathbf{A}_{m-1} = \mathbf{Q}_m\mathbf{R}_m + \rho\mathbf{I} \Rightarrow \mathbf{Q}_m^T\mathbf{A}_{m-1}\mathbf{Q}_m = \mathbf{R}_m\mathbf{Q}_m + \rho\mathbf{I} = \mathbf{A}_m$$