Remember to write your name! You are allowed to use a calculator. You are not allowed to use the textbook, your notes, the internet, or your neighbor. To receive full credit on a problem you must show sufficient justification for your conclusion unless explicitly stated otherwise.

Name:

1. (30 points) If the statement is always true mark “TRUE”; if it is possible for the statement to be false then mark “FALSE.” If the statement seems neither true nor false but rather incoherent, raise your hand. No justification is necessary. Students in 4720 can pick 5 out of 6 questions to answer. Students in 5720 must answer all.

   (a) If a matrix \( A \) is normal then the eigenvalues of the perturbed matrix \( A + E \) are all within a distance \( \|E\|_3 \) of the eigenvalues of \( A \).

   (b) Let \( \lambda \) be a simple eigenvalue of \( A \) with left and right eigenvectors \( y \) and \( x \), each of which are 2-norm unit vectors, and let \( A + E \) be the perturbed matrix where \( \|E\|_2 = \epsilon \). True or False: The perturbed matrix will have an eigenvalue \( \mu \) within a distance of approximately \( \epsilon / |y^*x| \) from \( \lambda \), for small-enough \( \epsilon \).

   (c) Let \( A \) be a diagonalizable matrix with approximate eigenvalue/eigenvector pair \( (\mu, x) \). True or false: \( (\mu, x) \) is an exact eigenvalue/eigenvector pair for a perturbed matrix \( A + E \) where \( \|E\|_2 \leq \|Ax - \mu x\|_2 \).

   (d) Suppose that \( A \) is \( n \times n \) with LU factorization \( PA = LU \). True or false: The matrix \( U P^T L \) has the same eigenvalues as \( A \).

   (e) Let \( \lambda \) and \( x \) be an eigenvalue/eigenvector pair for \( A \). True or false: The matrix \( A - \lambda xx^*/\| x \|_2^2 \) has eigenvector \( x \) with eigenvalue 0.

   (f) Let \( S \) be a nontrivial subspace that is invariant under a square matrix \( A \). True or False: There is an eigenvector of \( A \) in \( S \).

2. (20 points) Suppose that you are given one eigenvalue/eigenvector pair of an \( n \times n \) matrix \( A \). Explain how you can reduce the problem of finding the remaining eigenvalues of \( A \) to finding the eigenvalues of an \( n - 1 \times n - 1 \) matrix. Show explicitly how to construct the \( n - 1 \times n - 1 \) matrix. Hint: Start by constructing an invertible matrix \( X \) whose first column is the eigenvector.
3. (30 points) Computing the SVD of a real $m \times n$ matrix $A$ requires computing the eigenvalues and eigenvectors of $A^T A$ and $AA^T$.

(a) Let $P$ and $Q$ be real orthogonal matrices of size $m \times m$ and $n \times n$ respectively, and let $B = PAQ$. Show that the singular values of $B$ are the same as the singular values of $A$.

(b) Let $v$ be an eigenvector of $B^T B$. How is it related to the corresponding eigenvector of $A^T A$?

(c) It is possible to choose $P$ and $Q$ such that $B$ is bi-diagonal (nonzeros immediately above the diagonal). Prove that $B^T B$ and $BB^T$ are tridiagonal (you may cite any relevant theorem from class).

4. (20 points)

- **5720 Only** Let $A$ be an $n \times n$ diagonalizable matrix with eigenvalues satisfying $\lambda_1 = \ldots = \lambda_k$ with $|\lambda_k| > |\lambda_{k+1}| \geq \ldots \geq |\lambda_n|$. Show that the vectors generated by the power method will converge to an eigenvector of $A$ (under standard assumptions on the starting vector).

- **4720 Only** The basic shifted QR algorithm is

$$A_{m-1} - \rho I = Q_m R_m, \quad A_m = R_m Q_m + \rho I, \quad A_0 = A.$$  

Show that $A_m$ is orthogonally similar to $A_{m-1}$ (you may assume everything is real).