Remember to write your name! You are allowed to use a calculator. You are not allowed to use the textbook, your notes, the internet, or your neighbor. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

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1. (30 points) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." If the statement seems neither true nor false but rather incoherent, raise your hand. No justification is necessary. **Students in 4720 can pick 5 out of 6 questions to answer. Students in 5720 must answer all.**

_____(a) If a matrix $\mathbf{A}$ is normal then the eigenvalues of the perturbed matrix $\mathbf{A}+\mathbf{E}$ are all within a distance $\|\mathbf{E}\|_3$ of the eigenvalues of $\mathbf{A}$.

_____(b) Let λ be a simple eigenvalue of $\mathbf{A}$ with left and right eigenvectors $\boldsymbol{y}$ and $\boldsymbol{x}$, each of which are 2-norm unit vectors, and let $\mathbf{A}+\mathbf{E}$ be the perturbed matrix where $\|\mathbf{E}\|_2=\epsilon$. True or False: The perturbed matrix will have an eigenvalue μ within a distance of approximately $\epsilon/|\boldsymbol{y}^*\boldsymbol{x}|$ from λ, for small-enough ϵ.

_____(c) Let $\mathbf{A}$ be a diagonalizable matrix with approximate eigenvalue/eigenvector pair $(\mu,\boldsymbol{x})$. True or false: $(\mu,\boldsymbol{x})$ is an exact eigenvalue/eigenvector pair for a perturbed matrix $\mathbf{A}+\mathbf{E}$ where $\|\mathbf{E}\|_2\le\|\mathbf{A}\boldsymbol{x}-\mu\boldsymbol{x}\|_2$.

_____(d) Suppose that $\mathbf{A}$ is $n\times n$ with LU factorization $\mathbf{PA}=\mathbf{LU}$. True or false: The matrix $\mathbf{UP}^T\mathbf{L}$ has the same eigenvalues as $\mathbf{A}$.

_____(e) Let λ and $\boldsymbol{x}$ be an eigenvalue/eigenvector pair for $\mathbf{A}$. True or false: The matrix $\mathbf{A}-\lambda\boldsymbol{x}\boldsymbol{x}^*/\|\boldsymbol{x}\|_2^2$ has eigenvector $\boldsymbol{x}$ with eigenvalue 0.

_____(f) Let $\mathcal{S}$ be a nontrivial subspace that is invariant under a square matrix $\mathbf{A}$. True or False: There is an eigenvector of $\mathbf{A}$ in $\mathcal{S}$.

2. (20 points) Suppose that you are given one eigenvalue/eigenvector pair of an $n\times n$ matrix $\mathbf{A}$. Explain how you can reduce the problem of finding the remaining eigenvalues of $\mathbf{A}$ to finding the eigenvalues of an $n-1\times n-1$ matrix. Show explicitly how to construct the $n-1\times n-1$ matrix. Hint: Start by constructing an invertible matrix $\mathbf{X}$ whose first column is the eigenvector.

3. (30 points) Computing the SVD of a real $m \times n$ matrix $\mathbf{A}$ requires computing the eigenvalues and eigenvectors of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$.

   (a) Let $\mathbf{P}$ and $\mathbf{Q}$ be real orthogonal matrices of size $m \times m$ and $n \times n$ respectively, and let $\mathbf{B} = \mathbf{PAQ}$. Show that the singular values of $\mathbf{B}$ are the same as the singular values of $\mathbf{A}$.

   (b) Let $\boldsymbol{v}$ be an eigenvector of $\mathbf{B}^T\mathbf{B}$. How is it related to the corresponding eigenvector of $\mathbf{A}^T\mathbf{A}$?

   (c) It is possible to choose $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{B}$ is bi-diagonal (nonzeros immediately above the diagonal). Prove that $\mathbf{B}^T\mathbf{B}$ and $\mathbf{B}\mathbf{B}^T$ are tridiagonal (you may cite any relevant theorem from class).

4. (20 points)

   - **5720 Only** Let $\mathbf{A}$ be an $n \times n$ diagonalizable matrix with eigenvalues satisfying $\lambda_1 = \ldots = \lambda_k$ with $|\lambda_k| > |\lambda_{k+1}| \geq \ldots \geq |\lambda_n|$. Show that the vectors generated by the power method will converge to an eigenvector of $\mathbf{A}$ (under standard assumptions on the starting vector).
   - **4720 Only** The basic shifted QR algorithm is

   $$\mathbf{A}_{m-1} - \rho\mathbf{I} = \mathbf{Q}_m\mathbf{R}_m, \quad \mathbf{A}_m = \mathbf{R}_m\mathbf{Q}_m + \rho\mathbf{I}, \quad \mathbf{A}_0 = \mathbf{A}.$$

   Show that $\mathbf{A}_m$ is orthogonally similar to $\mathbf{A}_{m-1}$ (you may assume everything is real).