APPM 4/5560 Markov Chains

Fall 2019, Some Review Problems for Exam II

1. State and prove the lack of memory property of the exponential random variable.

2. Let $\{N(t)\}$ be a Poisson process with rate 3.

   (a) Find $P(N(2) = 3, N(6) = 3)$.

   (b) Find the expected number of arrivals between time 2 and time 3 given that there was already an arrival by time 2.

   (c) Find $P(N(2) = 3|S_2 = 1.8)$ where S_2 is the time of the second arrival.

3. A hitchhiker approaches a major interstate at a certain place and time of day where vehicles are known to pass according to a Poisson process with a rate of 200 cars per hour.

   (a) If each car here will pick up the hitchhiker with probability 0.05, what is the hitchhiker's expected waiting time?

   (b) What is the expected number of cars that will pass the hitchhiker before he/she is picked up?

   (c) Now suppose that 1 in 100 cars that pass this point are highway patrol cars and if each patrol car will stop to question/warn/ticket/arrest the hitchhicker with probablity 0.7, what is the probability that the hitchhiker will get a ride before being bothered by the highway patrol?

4. Let $X_1, X_2 \overset{iid}{\sim} exp(rate = \lambda)$. Find the pdf for $Y = \max\{X_1, X_2\}$.

5. Customers arrive at a bank according to a Poisson process with rate λ.

   (a) Suppose exactly one customer arrived during the first hour. What is the probability that he/she arrived during the first 20 minutes?

   (b) Suppose that exactly two customers arrived during the first hour. What is the probability that exactly one had arrived by 20 minutes?

   (c) Suppose that exactly two customers arrived during the first hour. What is the probability that at least one arrived in the first 20 minutes?

6. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Find the condition distribution of X given that $X + Y = n$.

7. Let $\{N(t)\}$ be a Poisson process with rate λ. For any $s, t \geq 0$ find

$$\mathsf{E}[N(t)N(s + t)].$$

8. Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate λ_1, and that, independently, major defects are distributed over the cable according to a Poisson process with rate λ_2. Let $N(t)$ be the number of defects in the first t feet of the cable.

(a) If exactly 1 minor defect has been found in the first 10 feet of the cable, what is the probability it is within the first 2 feet?

(b) If exactly 1 defect has been found in the first 10 feet of the cable, what is the probability that it is a minor defect?

(c) What is the expected number of minor defects between two successive minor defects?

9. Customers arrive at a service facility according to a Poisson process with rate λ customers per hour. Let $N(t)$ be the number of customers that have arrives up to time t. Let $S_1, S_2, \ldots$ be the successive arrival times. Find

   (a) the probability that 5 customers arrived during the first hour given that 12 arrived during the first 2 hours

   (b) $\mathsf{E}[S_5|N(1) = 3]$

   (c) $\mathsf{E}[S_5|S_1 > t]$

   (d) $\mathsf{E}[N(t)|S_1 > t]$

   (e) $\mathsf{E}[N(t)|S_1 < t]$

10. The gamma density can not be integrated in closed form. For $X \sim \Gamma(n, \lambda)$, find an expression for $P(X \leq x)$ in terms of a sum of Poisson probabilities.

11. Consider a two-server queueing process where customers arrive according to a Poisson process with rate λ, and go to either one of two servers if they are available. If both servers are busy, the customers form a single queue where the person at the end of the line will go to the next available server.

    Suppose that each sever takes an exponential amount of time with rate μ to serve a customer.

    If we model the total number of customers in this system as a birth-and-death process, what are the birth and death rates?

12. Consider a Poisson process $\{N(t)\}$ with rate λ with rateλ for some $\lambda > 0$.

    Does this process have a stationary distribution? If so, find it. If not, explain why it does not.

13. Suppose that d particles are distributed into two cells. A particle in cell 1 remains in that cell for a random length of time that is exponentially distributed with rate λ before moving to cell 2. A particle in cell 2 remains in that cell for a random length of time that is exponentially distributed with rate μ before moving to cell 1. The particles act independently of each other. Let $X(t)$ denote the number of paticles in cell 2 at time $t \geq 0$.

    Show that $X(t)$ is a birth-and-death process and find the birth-and-death rates.

14. (a) Suppose that f and g are both $o(h)$ functions. Show that f/g is $o(h)$ or give a counterexample if it is not true.

    (b) Simplify

    $$\frac{1 - e^{-\lambda h}}{h}$$

    with $o(h)$ notation.

(c) Show that $(1 - \lambda h)^i = 1 - i\lambda h + o(h)$.

15. A Poisson bug race: Two bugs, bug 1 and bug 2, are sitting on a 1 dimensional grid. After an exponential amount of time with rate λ_1, bug 1 hops forward 1 unit on the grid. Similarly, after an exponential amount of time with rate λ_2, bug 2 hops forward 1 unit on the grid. The first bug to hop 10 units wins.

    (a) What is the probability that bug 1 is the first to hop?

    (b) What is the probability that, after 4 total hops, the bugs are tied in the race?

    (c) What is the probability that bug 1 wins the race? (Leave your answered as an unsimplified sum.)

16. A small motel reservation system has two computers– one online and one backup.

    (a) The motel only operates one at a time. The operating computer fails after an exponentially distributed amount of time with rate μ and is replaced by the other computer if it is in operating condition. There is one repair facility (which can repair only one computer at a time) and repair times are exponentially distributed with rate λ. Let $X(t)$ be the number of computers in operating condition at time t. $\{X(t)\}$ is a Markov process. (You do not have to show this.) Find the infinitessimal generator matrix.

    (b) Find the generator matrix now assuming that both machines are operating and used at the same time.

17. Let $\{Y_n\}_{n=0}^{\infty}$ be a discrete time Markov chain on the state space $S = \{1, 2, 3, 4, 5\}$ with transition probabilities p_{ij} for $i, j \in S$. Let $\{N(t)\}_{t \geq 0}$ be an independent Poisson process with rate λ. Define a new "compound process" as

$$X(t) = Y_{N(t)}.$$

    Argue that $\{X(t)\}$ is a continuous time Markov chain and gives its infinitessimal generator matrix.