APPM 5720

Review Problems for In-Class Part of Exam I

1. Let X_1, X_2 be a random sample from the $N(0, \sigma^2)$ distribution. Find the distribution of

$$\frac{X_1}{|X_2|}?$$

Name it! (*Hint: Don't make this too hard!*)

2. Let T be a random variable with the t-distribution with n degrees of freedom.

   (a) What is the expected value of T? (This does not require any computation.)

   (b) Find the variance of T. (Hint: Do not use the pdf for T. You do not need to know the pdf for this exam. Instead, use the definition of T as a ratio of other random variables.)

3. Suppose that a random sample of size 10, taken from the $N(\mu, 3)$ distribution, results in a sample mean of 4.9. Give an 80% confidence interval for the true mean μ.

4. Let $X \sim unif(0, \theta)$. Give a 95% confidence interval for θ.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with pdf

$$f(x; \theta) = \frac{2}{\theta^2} x e^{-(x/\theta)^2} \, I_{(0,\infty)}(x)$$

(Note: This is a "Weibull" distribution.)

   (a) Show that $W = 2\sum_{i=1}^n X_i^2/\theta^2 \sim \chi^2(2n)$.

   (b) Use W to derive a $100(1-\alpha)\%$ confidence interval for θ. (Your answer should involve χ^2 critical values. ie: Use the notation $\chi^2_{\alpha,n}$ to denote the value that cuts off area α to the right on a $\chi^2(n)$ curve.)

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with rate λ. Consider testing the hypotheses

$$\begin{array}{lcl} H_0 & : & \lambda = \lambda_0 \\ H_1 & : & \lambda < \lambda_0 \end{array}$$

   (a) Find a test of size α based on the $X_{(1)}$, the minimum value in the sample.

   (b) Find the power function for your test from part (a).

7. Let X be a random sample of size 1 from the shifted exponential distribution with rate 1 which has pdf

$$f(x; \theta) = e^{-(x-\theta)} \, I_{(\theta,\infty)}(x).$$

   (a) Find a test of size α for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ based on looking at that single value in the sample.

   (b) Find the power function for your test.

   (c) Find a test of size α for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ based on the minimum of a random sample of size n from this distribution.

(d) Compute the power function for your test from part (c). Compare it to the power function from part (b). Based on these power functions alone, decide which test is better. Explain.

8. Consider a random sample of size n from the uniform$(0, \theta)$ distribution.

   (a) Find the probability that the random interval $(X_{(n)}, 2X_{(n)})$ contains θ.

   (b) Find the constant c such that $(X_{(n)}, cX_{(n)})$ is a $100(1-\alpha)\%$ confidence interval for θ.

9. Let $X_1, X_2, \dots, X_n$ be a random sample from the $N(\mu, \sigma^2)$ distribution. Recall that $\overline{X} \sim N(\mu, \sigma^2)$ and $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$ are independent.

   (a) Using the definition of the t-distribution as a ratio of random variables, find the distribution of $\frac{\overline{X}-\mu}{S/\sqrt{n}}$.

   (b) Use part (a) to derive a $100(1-\alpha)\%$ confidence interval for μ from scratch.

   (c) Derive a $100(1-\alpha)\%$ confidence interval for σ^2.

10. Let $X_1, X_2, \dots, X_{16}$ be a random sample from the $N(\mu, 9)$ distribution. Find a test of size 0.05 of

$$\begin{aligned} H_0 &: \mu = 3 \\ H_1 &: \mu \neq 3 \end{aligned}$$

    based on the sample mean $\overline{X}$.

11. Let $X_1, X_2, \dots, X_n$ be a random sample from the Weibull distribution with parameters α and β, with γ fixed and known to be 1. Find method of moments estimators for α and β.

12. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with pdf

$$f(x; \theta) = \frac{3}{\theta^3} x^2 \cdot I_{(0,\theta)}(x)$$

    where $\theta > 0$. Find the method of moments estimator (MME) of θ.

13. Let $X_1, X_2, \dots, X_n$ be a random sample from the Pareto distribution with parameter γ. Find the maximum likelihood estimator (MLE) of γ.

14. Let $X_1, X_2, \dots, X_n$ be a random sample from the uniform distribution over the interval $(0, \theta)$ for some $\theta > 0$.

    (a) Find the maximum likelihood estimator (MLE) of θ.

    (b) Find an MLE for the median of the distribution. (The median is the number that cuts the area under the pdf exactly in half.)

15. Let $X_1, X_2, \dots X_n$ be a random sample from the distribution with pdf

$$f(x; \theta, \lambda) = \theta \, \lambda^\theta \, x^{-(\theta+1)} \, I_{(\lambda,\infty)}(x)$$

    where $\theta > 0$ and $-\infty < \lambda < \infty$.

    Find the MLEs for θ and λ.

16. Let $X_1, X_2, \ldots, X_n$ be a random sample from the shifted exp onential distribution with rate 1 which has pdf

$$f(x;\theta) = e^{-(x-\theta)}\, I_{(\theta,\infty)}(x).$$

   (a) Find the MME of θ.
   (b) Find the MLE of θ.
   (c) To simplify further computations, show that $\overline{X} - \theta \sim GAMMA(n, n)$ and that $X_{(1)} - \theta \sim exp(rate = n)$. (Hint: Personally, I think it is easiest to use mgf's for the first problem and cdf's for the second.)
   (d) Are your two estimators unbiased? (Hint: Use part (c). For example, if you are trying to find $\mathsf{E}[X_{(1)}]$, you could write this as $\mathsf{E}[X_{(1)} - \eta + \eta] = \mathsf{E}[X_{(1)} - \eta] + \eta$.)
   (e) Find the MSE for each of your two estimators.
   (f) At this point, you should have found that your MME was unbiased and that your MLE was biased. Can you add or subtract something from your biased MLE to make an unbiased estimator, $\hat{\theta}_3$, for θ? Find the MSE for this new estimator and decide which is the best (with respect to MSE) estimator of all three.

17. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on the interval $(0, \theta)$. Consider testing

$$\begin{array}{rcl} H_0 & : & \theta = \theta_0 \\ H_1 & : & \theta < \theta_0 \end{array}$$

   (a) Find a test of size α based on the $X_{(n)}$, the maximum of the sample.
   (b) Find a test of size α based on the $X_{(1)}$, the minimum of the sample.

18. Let $X_1, X_2, \ldots X_n$ be a random sample from the binomial distribution with parameters m and p with m fixed and known. Find the MLE of $P(X = 0)$.

19. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta(1+x)^{-(1+\theta)} I_{(0,\infty)}(x), \qquad \theta > 0.$$

   (a) Find the MLE $\hat{\theta}_n$ of θ.
   (b) What is $\lim\limits_{n\to\infty} \mathsf{E}[\hat{\theta}_n]$?

20. Let $X_1, X_2, \ldots, X_n$ be a random sample from the geometric distribution

$$f(x;p) = p(1-p)^x I_{\{0,1,\ldots\}}(x), \qquad 0 < p < 1.$$

   (a) Find the MME for p.
   (b) Find the MLE for p.
   (c) Find the MLE for $\ln p$.
   (d) Find the asymptotic distribution of $\hat{p}$ (MLE).

21. Consider a random sample of size n from the distribution with pdf

$$f(x;\theta) = \frac{(\ln\theta)^x}{\theta x!} \cdot I_{\{0,1,\ldots\}}$$

with $\theta > 1$.

(a) Find the CRLB for θ.

(b) Find the CRLB for $(\ln \theta)^2$.

(c) Find the CRLB for $P(X_1 \leq 1)$.

22. (a) Show that $\mathsf{E}\left[\frac{\partial}{\partial\theta} \ln f(\vec{X};\theta)\right] = 0$.

    (b) What is $Var\left[\frac{\partial}{\partial\theta} \ln f(\vec{X};\theta)\right]$ in terms of the Fisher Information?

23. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential rate θ distribution.

    (a) Find the CRLB for θ. What is it bounding?

    (b) Find the CRLB for $1/\theta$.

    (c) Find the MLE of $P(X > 1)$.

    (d) Find the CRLB for $P(X > 1)$.

24. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta(1+x)^{-(1+\theta)} I_{(0,\infty)}(x), \qquad \theta > 0.$$

Find the asymptotic distribution of $\hat{\theta}_n$, where $\hat{\theta}_n$ is the MLE for θ.