APPM 5560 Markov Chains

Fall 2019 Exam Two, Take Home Part

Due Monday, April 22nd, Plus or Minus...

Welcome to the take-home part of exam II. This is an exam, so please do not discuss it with anyone. Except me– you are more than welcome to come talk to me about it!

1. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Let N be a positive integer-valued random variable that is independent of the X_i sequence. We have already seen that

$$\mathsf{E}\left[\sum_{i=1}^{N} X_i\right] = \mathsf{E}[N] \cdot \mathsf{E}[X_1].$$

   Find and prove a similar formula for the variance of $\sum_{i=1}^{N} X_i$.

2. Starting at time 0, satellites are launched at times of a Poisson process with rate λ. Suppose that each satellite, once launched, has a lifetime, indepedent of all others and of the lauch process, that has cdf F and mean μ. Let $X(t)$ be the number of launched and working satellites at time t.

   (a) Find the distribution of $X(t)$.

   (b) Let $t \to \infty$ to show that the limiting distribution is Poisson($\lambda\mu$).

3. Consider the following *random telegraph signal*

$$X(t) = (-1)^{N(t)}\, Y$$

   for $t \geq 0$, where $\{N(t)\}$ is a homogeneous Poisson process with rate λ and Y is a binary random variable that takes on the values ± 1 with probability $1/2$ each. Assume that Y is independent of $\{N(t)\}$.

   (*Hint: For the following questions, you may find your work is more streamlined if you recall/use thehyperbolic trigonometric functions* $\sinh x = \frac{1}{2}(e^x - e^{-x})$ *and* $\cosh x = \frac{1}{2}(e^x + e^{-x})$.)

   (a) Let $M(t) = (-1)^{N(t)}$. Find closed form expressions for $P(M(t) = 1)$ and $P(M(t) = -1)$.

   (b) Find $\mathsf{E}[M(t)]$.

   (c) For $s < t$, let $p_{ij} = P(M(s) = i, M(t) = j)$. Find p_{11}, p_{-11}, $p_{1,-1}$, and $p_{-1,-1}$.

   (d) Find $\mathsf{E}[M(s)M(t)]$ for any $s, t > 0$. (*Hint: Focus first on the case where $s < t$ and then generalize.*)

   (e) Find $Cov(X(s), X(t))$.