APPM 5470, Methods of Applied Mathematics: Partial Differential Equations and Integral Equations, Exam 2, 11/6 *Fall 2017*

Answer all the questions and show your work/reasoning. Justify your answers. Partial credit will be given.

1. **Heat equation:** Consider the Neumann problem for the inhomogeneous heat equation on a disk

$$
\begin{aligned}
&u_t(\mathbf{x},t) = \Delta u(\mathbf{x},t) + F(\mathbf{x},t), \quad \mathbf{x} \in B(0,1) \subset \mathbb{R}^2, \quad t > 0, \\
&\frac{\partial u}{\partial n}(\mathbf{y},t) = g(\mathbf{y},t), \quad \mathbf{y} \in \partial B(0,1), \quad t > 0, \\
&u(\mathbf{x},0) = f(\mathbf{x}), \quad \mathbf{x} \in B(0,1),
\end{aligned}
\tag{1}
$$

where f, g, and F are given functions and $\mathbf{n}(\mathbf{y})$ is the outward unit normal to the boundary of the disk.

   (a) [20 pts] Suppose $g(\mathbf{y},t) = 0$ and $F(\mathbf{x},t) = 0$ over their respective domains. Is the energy $E_1(t) = \int_{B(0,1)} u(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$ conserved? Is the energy $E_2(t) = \frac{1}{2}\int_{B(0,1)} u(\mathbf{x},t)^2\,\mathrm{d}\mathbf{x}$ conserved? Justify your answers.

   (b) [20 pts] Assuming the existence of a solution to the problem (1), use an energy argument to prove that the solution is unique.

   (c) [10 pts] Suppose $F(\mathbf{x},t) < 0$ and fix any $T > 0$. Prove that the solution $u(\mathbf{x},t)$ of problem (1) cannot attain a local maximum at any point in the parabolic cylinder $(\mathbf{x},t) \in U_T = B(0,1) \times (0,T]$.

2. **Separation of variables:** Consider the following initial-boundary value problem for the heat equation

$$
\begin{aligned}
u_t &= u_{xx}, \quad x \in (0,\pi), \quad t > 0, \\
u(0,t) &= 0, \quad u_x(\pi,t) = 0, \quad t > 0, \\
u(x,0) &= f(x), \quad x \in (0,\pi).
\end{aligned}
\tag{2}
$$

   (a) [30 pts] Find a formal series solution and provide integral formulas for the coefficients.

   (b) [20 pts] Suppose $f \in C^2[0,\pi]$ and $f(0) = f(\pi) = 0$. Prove that the formal solution is a classical solution, i.e., that it satisfies all the conditions of problem (2).