APPM 4360/5360 Introduction to Complex Variables and Applications

EXAM #1 Tuesday February 19, 2019

A formulae sheet is allowed: 8.5×11 **inches; NO calculators, phones, computers allowed; explain all work**

XC: Extra Credit

1. (12 points) Given the complex numbers:

a) $e^{3\pi i/2}$ b)$e^{4\pi i/3}$ Put in Cartesian coordinate form: $x + iy$

c) -1 d) $\frac{1}{1-i}$ Put in polar coordinate form: $re^{i\theta}, 0 \leq \theta < 2\pi$

Solution:

a) $e^{3\pi i/2} = \cos(3\pi/2) + i\sin(3\pi/2) = -i$.

b) $e^{4\pi i/3} = \cos(4\pi/3) + i\sin(4\pi/3) = -1/2 - i\sqrt{3}/2$.

c) $-1 = e^{i\pi}$.

d)

$$\frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{2} = re^{i\theta},$$

then $r^2 = 1/2$ and $r = 1/\sqrt{2}$; $x = y = 1/2 > 0$ so $\theta = \pi/4$.

$$\frac{1}{1-i} = \frac{1}{\sqrt{2}}e^{i\pi/4}.$$

2. (14 points) Evaluate the following limits; simplify answer:

a) $\lim_{z\to i} \frac{e^{\pi z}+1}{z-i}$ b) $\lim_{z\to\infty} \frac{z+1}{e^z}$

Solution:

a)

$$\lim_{z\to i} \frac{e^{\pi z}+1}{z-i} = \lim_{z\to i} \frac{e^{\pi i}e^{\pi(z-i)}+1}{z-i} = \lim_{z\to i} \frac{-(1+\pi(z-i)+(\pi(z-i))^2/2+\dots)+1}{z-i} =$$

$$= \lim_{z\to i} \frac{-\pi(z-i)-(\pi(z-i))^2/2+\dots}{z-i} = \lim_{z\to i}(-\pi - \pi(z-i) + \dots) = -\pi.$$

b) $\lim_{z\to\infty} \frac{z+1}{e^z}$ does not exist. Consider two different ways of approaching ∞: one is $z \to +\infty$ on positive real axis, then the limit is 0; the other is $z \to -\infty$ on negative real axis, then the limit is $-\infty$. Thus, we got two different values which proves the statement.

3. (9 points) Suppose both $f(z)$ and $\overline{f(z)}$ are analytic in a domain D. Deduce/explain why this function must be a constant.

Solution:

Let $f(z) = u + iv$, u and v real, then $\overline{f(z)} = u - iv$. Analyticity implies CR conditions. For $f(z)$ they are

$$u_x = v_y, \qquad v_x = -u_y,$$

and for $\overline{f(z)}$ they are

$$u_x = -v_y, \qquad v_x = u_y.$$

Since both functions are analytic, all four CR conditions must be satisfied. This implies $u_x = v_y = v_x = u_y = 0$ i.e. both u and v are constant. So $f(z)$ is also constant.

4. (15 points) Explain where the functions are analytic or have singular points. If they have singular points find where the singular points are located.

a) $z/(z^2 + 2z + 1)$ b) $z^5 + \sqrt{z}$ c) $z^2/\sin z$

Solution:

a) $z/(z^2 + 2z + 1)$: this is a rational function, it is analytic everywhere except for the zeros of its denominator, i.e. except for z s.t. $z^2 + 2z + 1 = (z + 1)^2 = 0$. Thus, the only singular point is $z = -1$.

b) $z^5 + \sqrt{z}$: this is multivalued function with branch points $z = 0$ and $z = \infty$. Each of its two branches is analytic outside of the branch cut which must connect 0 and ∞.

c) $z^2/\sin z$: this is a ratio of two entire functions; it is analytic everywhere except for the zeros of its denominator, i.e. except for z s.t. $\sin z = 0$. Thus the singular points are $z = n\pi$ for all $n \in \mathbb{Z}$.

(in fact $z = 0$ is special s.p. – removable; the function can be redefined there so that it is not an s.p. anymore.)

5. (9 points) Suppose we are given the complex potential:

$$\Omega(z) = \log z$$

Find the velocity potential, the stream function and velocity field. Sketch the flow.

Solution:

Let $\Omega(r,\theta) = \phi(r,\theta) + i\psi(r,\theta)$. Since $\log z = \log r + i\theta$, where $r = |z|$ and θ is the angle between the line connecting 0 and z and positive x direction. Then the velocity potential $\phi = \log r$ and the stream function $\psi = \theta$, where $r = |z| = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. For the components of the velocity field V we get

$$V_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}, \qquad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0,$$

so we have only nonzero V_r component which means that the velocity is purely radial relative to the point $z = 0$ and $\mathrm{sign}(V_r) = +$ means it points away from 0. The streamlines (flow lines) are rays emanating from $z = 0$.

6. (18 points) a) (9 points) Find all branch points and how many Riemann sheets are associated with the following functions: i) $\log(z+i)$ ii) $1/z^{1/3}$ iii) $1/z$

b) (9 points) Find all branch points and give/explain how to find a branch cut associated with the function: $f(z) = (\frac{z}{z-2})^{1/2}$

Solution:

a) i) $\log(z+i)$: this is a simple log function, so the branch points are $z = -i$ and $z = \infty$; there are infinitely many Riemann sheets (different branches) associated with it.
ii) $1/z^{1/3}$: this is a power function $z^{-1/3}$, so the branch points are $z = 0$ and $z = \infty$; the power is rational $m/l = -1/3$, $l = 3$, so there are three Riemann sheets (different branches) associated with it.
iii) $1/z$: this is a single-valued (rational) function analytic everywhere except for $z = 0$, its singular point. There are no branch points and one Riemann sheet (complex plane or Riemann sphere).

b)

$$f(z) = \left(\frac{z}{z-2}\right)^{1/2}.$$

This is a rational function singular at $z=2$, taken to the power of $1/2$. Therefore the branch points are those where

$$\frac{z}{z-2}=0 \qquad \text{or} \qquad \frac{z}{z-2}=\infty,$$

i.e. $z=0$ and $z=2$ ($z=\infty$ is not a b.p.). A branch cut must connect the two branch points, the simplest one is the interval $[0,2]\in\mathbb{R}$. To confirm this, consider principal angles θ_1, θ_2 s.t.

$$z=r_1e^{i\theta_1}, \qquad z-2=r_2e^{i\theta_2}, \qquad \Longrightarrow \qquad \left(\frac{z}{z-2}\right)^{1/2}=re^{i\Theta}=\left(\frac{r_1}{r_2}\right)^{1/2}e^{i(\theta_1-\theta_2)/2},$$

and the angle ranges are

$$0\le\theta_1\le 2\pi, \qquad 0\le\theta_2\le 2\pi.$$

Then we have (at the top and bottom of x-axis, see pictures in section 2.3 of the textbook)

θ_1	θ_2	Θ	Region
0	0	0	$\{(x,y)\|x>2,y>0\}$
0	π	$-\frac{\pi}{2}$	$\{(x,y)\|0<x<2,y>0\}$
π	π	0	$\{(x,y)\|x<0,y>0\}$
π	π	0	$\{(x,y)\|x<0,y<0\}$
2π	π	$\frac{\pi}{2}$	$\{(x,y)\|0<x<2,y<0\}\}$
2π	2π	0	$\{(x,y)\|x>2,y<0\}$

Here the value of $e^{i\Theta}$ is important (not just Θ) but this still has discontinuity on $[0,2]$ which then becomes the location of the cut.

7. (14 points) Find the integrals below

a) $\int_C(z+1/z+1/z^2)dz$ where C is the unit circle $|z|=1$

b) $\int_C \bar{z}dz$ where C is the unit circle $|z|=1$

Solution:

a)

$$\int_C(z+1/z+1/z^2)dz=\int_C zdz+\int_C\frac{dz}{z}+\int_C\frac{dz}{z^2},$$

and, using the formula for integrating the powers of z over C, one gets

$$\int_C(z+1/z+1/z^2)dz=0+2\pi i+0=2\pi i.$$

b) $z = e^{i\theta}$ and $\bar{z} = e^{-i\theta}$ on C, so

$$\int_C \bar{z} dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta} d\theta = 2\pi i.$$

8. (9 points) Consider the integral $\int_C f(z)dz$. Let $f(z)$ be a continuous function of z on C where C is a square with center at the origin and a corner at $z = a + ia$. Find an upper bound for the absolute value: $|\int_C f(z)dz|$, of this integral; explain.

Solution:

Since $f(z)$ is continuous on C, which is a closed bounded region, it is also bounded there. Then let $M \geq 0$ be an upper bound of $|f(z)|$ on C. Use ML-inequality to get

$$|\int_C f(z)dz| \leq \int_C |f(z)||dz| \leq M \int_C |dz| = ML(C) = M \cdot 2a \cdot 4 = 8Ma.$$

XC (10 points)

Consider the integral:

$$I(\epsilon) = \int_{C_\epsilon} \frac{e^z}{z^{1/2}} dz$$

with $z^{1/2}$ having a branch cut from 0 to ∞ where C_ϵ is a punctured circle centered at the origin with radius ϵ that does not go through the branch cut (the circle begins just above the branch cut and ends just below the branch cut). Find an upper bound for $|I(\epsilon)|$ and evaluate $\lim_{\epsilon \to 0} I(\epsilon)$; explain.

Solution:

On C_ϵ, we have $z = \epsilon e^{i\theta}$, $0 < \theta < 2\pi$. Then

$$|I(\epsilon)| = \left| \int_{C_\epsilon} \frac{e^z}{z^{1/2}} dz \right| \leq \int_{C_\epsilon} \frac{|e^z|}{|z|^{1/2}} |dz| = \int_0^{2\pi} \frac{|e^{\epsilon(\cos\theta + i\sin\theta)}|}{\epsilon^{1/2}} \epsilon d\theta =$$

$$= \int_0^{2\pi} e^{\epsilon \cos\theta} \epsilon^{1/2} d\theta \leq e^\epsilon \epsilon^{1/2} \int_0^{2\pi} d\theta = 2\pi e^\epsilon \epsilon^{1/2}.$$

Since

$$\lim_{\epsilon \to 0} e^\epsilon \epsilon^{1/2} = \lim_{\epsilon \to 0} \epsilon^{1/2} = 0,$$

we also have $\lim_{\epsilon \to 0} I(\epsilon) = 0$.