APPM 5720: Computational Bayesian Statistics

Exam I Review Problems

Exam I: In class on Friday, February 23rd.

1. Know how to define or describe the following types of priors: conjugate, natural conjugate, non-informative, expert, mixture, improper, Jeffreys

2. (a) State deFinetti's Theorem.

   (b) Suppose that $X_1, X_2, \dots$ is an infinite sequence of iid random variables. Further suppose that the distribution of X_i is an equally weighted mixture of a $Bernoulli(\theta_1)$ distribution and a $Bernoulli(\theta_2)$ distribution.

   (c) Find the cdf F from deFinetti's Theorem.

3. Suppose that X is a random variable that takes on the value 0 with probability 1/4, the value 1 with probability 1/4, and otherwise takes on a value drawn from a continuous exponential distribution with rate λ restricted to the interval $(0, 1)$.

   (a) Write down the cdf for X.

   (b) Compute $\mathsf{E}[X]$ by evaluating the Riemann-Stiltjes integral

   $$\mathsf{E}[X] = \int x\, dF(x).$$

4. Show that if $X_1, X_2, \dots, X_n$ are exchangeable, then

   $$Var\left[\sum_{i=1}^{n} X_i\right] = nVar[X_1] + n(n-1)Cov(X_1, X_2).$$

5. The number of offspring in a certain population has probability mass function

   $$f(x|\alpha, \beta) = \begin{cases} \alpha & , \quad x = 0 \\ (1-\alpha)\beta(1-\beta)^{x-1} & , \quad x = 1, 2, \dots. \end{cases}$$

   Suppose that α and β have *a priori* independent Beta distributions with parameters (a, b) and (c, d), respectively.

   (a) Find the posterior distribution for (α, β).

   (b) Are α and β *a posteriori* independent?

   (c) Is the prior a conjugate prior?

6. The continuous Pareto distribution on your table of distributions is a specific case of a more generalized two-parameter Pareto distribution that has pdf

   $$f(x|\alpha, \beta) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\, I_{(\beta,\infty)}(x).$$

   Let us denote this distribution as $Pareto(\alpha, \beta)$.

   Suppose that $X_1, X_2, \dots, X_n \stackrel{iid}{\sim} unif(0, \theta)$.

   If θ has a $Pareto(\alpha, \beta)$ prior (for known α and β), find the posterior distribution for θ. Is this a conjugate prior?

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with pdf

$$f(x|\theta, \beta) = \theta\beta^\theta x^{\theta-1}\, I_{(0,1/\beta)}(x)$$

with parameters $\theta > 0$, $\beta > 0$. Suppose that θ is fixed and known.

Find the Jeffreys prior for β.

8. Suppose that the lifetimes of lightbulbs (in hours) produced in a single production run at the ACME Lightbulb Company are iid random variables following an exponential distribution with rate θ. Suppose that θ varies from run to run according to a $\Gamma(\alpha, \beta)$ distribution for known α and β.

   (a) Find the posterior Bayes estimator for θ based on a sample of lightbulb lifetimes $X_1, X_2, \ldots, X_n$.

   (b) Find the posterior predictive density for X_{n+1}.

   (c) What is the posterior probability that the next lightbulb sampled works for more than 75 hours?

9. Suppose that $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} N(\mu, 1)$.

   (a) Assuming a "flat" prior on μ, find the posterior Bayes estimator for μ.

   (b) Give the natural conjugate prior for μ.

10. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the distribution with pdf

$$f(x|\theta) = e^{-(x-\theta)}\, I_{(\theta,\infty)}(x).$$

Assuming a flat prior, find a 95% credible interval for θ.