APPM 4/5560 Markov Processes

Fall 2019, Some Review Problems for Exam One

(Note: You will not have to invert any matrices on the exam and you will not be expected to solve systems of 3 or more unknowns. There are some problems like this on this review. If they appeared on the exam, they will either be super trivial or you will be directed to "set up but not solve" the system.)

1. One card is selected from a deck of 52 cards and is discarded. A second card is then selected from the remaning 51 cards.
   (a) What is the probability that the second card selected is an ace?
   (b) Given that an ace was drawn as the second card, what is the probability that the discarded card was an ace?

2. Thirteen cards numbered $1, 2, \ldots, 13$ are shuffled and dealt face up one at a time. We say that a *match* occurs if the kth card revealed is card number k. Let N be the total number of matches that occur in the thirteen cards. Find $\mathsf{E}[N]$. (*Write N as the sum of indicators $I_1, I_2, \ldots, I_{13}$, where $I_i = 1$ if the ith card dealt is a match.*)

3. A Markov chain $\{X_n\}$ on the state space $\{0, 1\}$ has transition probability matrix

$$\mathbf{P}_1 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.8 & 0.2 \\ 0.5 & 0.5 \end{array} \right\| \end{array}$$

   (a) If, initially, we are equally likely to start in either state, find $P(X_1 = 0)$.
   (b) Find $P(X_3 = 1 | X_0 = 0)$.
   (c) What is the long run proportion of visits the chain will make to state 0?

4. Find all the communication classes of the Markov chain with the given transition probability matrix and classify each class as either recurrent or transient. Is the chain irreducible?

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

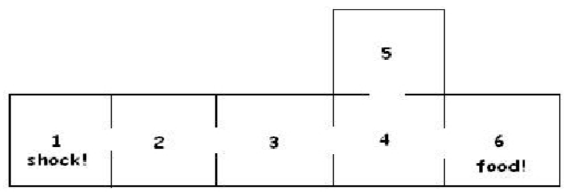
5. An urn contains 2 red balls and 1 yellow ball. At each point in time, a ball is selected at random from the urn and replaced with a yellow ball. The selection process continues until all of the red balls are removed from the urn. What is the expected duration of the process?

6. A rat with a bad cold (so he can't smell!) is put into compartment 4 of the maze shown in Figure 1. At each time step, the rat moves to another compartment. (He never stays where he is.) He chooses a departure door from each compartment at random. We want to know the probability that he finds the food before he gets shocked.

   (a) Set up an appropriate transition matrix and the system of equations you would need to solve this. **Do not solve the system!!!**

   (b) Is this system periodic? Aperiodic? Neither?

Figure 1: Rat Maze for Problem 4



7. Consider the Markov chain on $S = \{0, 1, 2\}$ running according to the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

   (a) What is the long run percentage of time that the chain is in state 1?

   (b) If the chain starts in state 0, what is the expected number of steps until the chain hits state 1?

   (c) If the chain starts in state 0, and if T is the first time that it hits either one of the states 1 or 2, what is the probability that the chain will be in state 2 at time T?

   (d) Starting in state 0, what is the mean time that the process spends in state 1 prior to first hitting state 2?

   (e) Starting in state 0, what is the mean time that the process spends in state 1 prior to returning to state 0?

   (f) Find $P(X_2 = 1 | X_2 \neq 2, X_1 \neq 2, X_0 = 0)$. Without computing it, do you expect $P(X_2 = 1 | X_1 \neq 2, X_0 = 0)$ to be smaller or larger? Explain.

8. Give an example of a transition probability matrix for a Markov chain having period 3.

9. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy.

(a) Set this problem up as a four state Markov chain.
(b) What is the probability that it will rain on Wednesday given that it did not rain on Sunday or Monday?
(c) Find the stationary distribution for this chain.

10. Let π be a stationary distribution of a Markov chain and let i and j be two states such that $\pi_i > 0$ and $i \to j$. Show that $\pi_j > 0$.

11. A Markov chain $\{X_n\}$ on the state space $\{1, 2\}$ has transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left\| \begin{array}{cc} 0.6 & 0.4 \\ 0.3 & 0.7 \end{array} \right\| \end{array}$$

(a) If, initially, we are equally likely to start in either state, find $P(X_2 = 2)$.
(b) What is the long run proportion of visits the chain will make to state 2?

12. An urn contains two red bugs and one green bug. Bugs are chosen at random, one by one, from the urn. If a red bug is chosen, it is removed. If the green bug is chosen, it is returned to the urn. The selection process continues until all of the red bugs are removed from the urn. What is the mean duration of the game?

13. Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

SET UP BUT <u>DO NOT SOLVE</u>, systems of equations to answer the following two questions. (Define variables used and specify what needs to be solved for.)

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.
(b) Determine the mean time to absorption for this Markov chain.

14. A Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ has a transition probability matrix given by

$$\mathbf{P}_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 \\ 1/6 & 1/2 & 0 & 1/6 & 1/6 & 0 \\ 1/4 & 0 & 1/3 & 5/24 & 5/24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

Find all communication classes and specify whether the states in each are transient or recurrent.

15. Consider the Markov chain on $S = \{0, 1, 2\}$ running according to the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

Specify how to find $P(X_5 = 2 | X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0)$.

(You need not actually compute the answer.)

16. For the Markov chain given in problem 13 above, SET UP BUT <u>DO NOT SOLVE</u> systems of equations to answer the following questions.

    (a) Find the stationary distribution.

    (b) If the chain starts in state 2, find the expected number of times the chain hits state 1 before hitting state 0.

17. On a remote and mysterious island, a sunny day is followed by another sunny day with probability 0.9, whereas a rainy day is followed by another rainy day with probability 0.2. Suppose that there are only sunny or rainy days. In the long run, what fraction of days are sunny?

18. Find all communication classes of the Markov chain with the given transition probability matrix and classify each state as either recurrent or transient. Is the chain irreducible?

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/8 & 3/8 & 0 \\ 1/3 & 0 & 1/6 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

19. In front of you are five urns which are painted red, green, yellow, and blue. The red urn contains 5 red balls, the green urn contains 3 green balls, the yellow urn contains 4 yellow balls and one ball each of red, green, and blue, and the blue urn contains one red, two green, and four yellow balls. A ball is drawn randomly from an urn, its color noted, and it is replaced in the urn. Then a ball is drawn from the urn matching the previous ball. This process is repeated indefinitely.

    (a) Set up the transition probability matrix for $\{X_n\}$ where X_n is the color of the urn drawn from on the nth trial.

    (b) Set up, but DO NOT SOLVE, a system of equations to answer the following question. If your first draw is from the yellow urn, what is the probability you eventually wind up in the red one? (Identify all variables used and indicate what you need to solve for.)

20. Consider the Markov chain on $S = \{0, 1, 2\}$ running according to the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

Specify how to find $P(X_5 = 2 | X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0)$.

(You need not actually compute the answer.)

21. A study of the strengths of Ivy League football teams shows that if a school has a strong team one year it is equally likely to have a strong team or average team next year; if it has an average team, half the time it is average next year, and if it changes it is just as likely to become strong as weak; if it is weak it has 2/3 probability of remaining so and 1/3 of becoming average.

    A school has a weak team. Set up BUT DO NOT SOLVE a Markov chain and a system of equations to determine the mean number of years until the school has a strong team. (Identify all variables used and indicate what you need to solve for.)

22. A two state Markov chain has transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array} \right\| \end{array}$$

    (a) Find the first return distribution $g_0^{(n)}$.
    (b) Use part (a), as opposed to a first step analysis to find, starting at 0, the mean number of steps it will take to return to 0.

23. Write down and interpret the Chapman-Kolmogorov equations.

24. Let $\{X_n\}$ be a Markov chain on the state space $S = \{0, 1, 2, 3, \ldots\}$. Which of the following are stopping times for the chain? Whenever one is not, give a brief explanation as to why it is not.

    (a) $T = \min\{n \geq 0 : X_n < 10\}$
    (b) $T =$ the third time X_n visits state 1
    (c) $T = \max\{n \geq 0 : X_n = 5\}$
    (d) Let $A = \{1, 3, 7\}$. Define $T = \max\{n \in A : X_n = 5\}$.
    (e) Let $A = \{1, 3, 7\}$. Define $T = \min\{n \in A : X_n = 5\}$.
    (f) $W = T - 1$ where T is defined in part (a).
    (g) the sum of two stopping times S and T

25. Consider a Markov chain on $S = \{0, 1, 2\}$ with the following transition probability matrix.

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array} .$$

Let $T_i = \min\{n \geq 1 : X_n = i\}$.

Verify that $\pi_i = 1/\mathsf{E}_i[T_i]$ for $i = 1, 2, 3$.