APPM 5560 Markov Chains

Fall 2019 Exam One, Take Home Part

Due Monday, March 4th

Welcome to the take-home part of exam I. This is an exam, so please do not discuss it with anyone. Except me– you are more than welcome to come talk to me about it!

1. A spider and a fly move between three possible locations on a web according to Markov chains $\{S_n\}$ and $\{F_n\}$, respectively, with transition probabilities

$$\mathbf{P}_S = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}, \qquad \mathbf{P}_F = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

At time $T = \min\{n : S_n = F_n\}$, the chase is over and the fly is caught. (Yeah, it makes no sense that the fly can move in the web, but this problem is too boring with the fly stuck in one place so just so just go with it!)

   (a) Suppose that $S_0 = i$ and $F_0 = j \neq i$. Find the expected value of T.

   (b) Suppose that the spider generalizes his strategy to

$$\mathbf{P}_S = \begin{Vmatrix} 1-2p & p & p \\ p & 1-2p & p \\ p & p & 1-2p \end{Vmatrix}$$

   What value of p should the spider choose to minimize the expected time to catch the fly?

2. Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed non-negative integer-valued random variables with

$$p_i = P(Y_n = i), \qquad \text{for } i = 0, 1, 2, \ldots$$

For $n = 1, 2, \ldots$, let

$$X_n = \max\{Y_1, Y_2, \ldots, Y_n\}.$$

   (a) Show that $\{X_n\}$ is a Markov chain.

   (b) Find the transition probability matrix for $\{X_n\}$.

   (c) Let $T = \min\{n \geq 1 : X_n \geq M\}$ where M is some fixed positive number. Let $\mu = \mathsf{E}[T]$. Use a first-step analysis to establish that

$$\mu = 1 + \mu P(Y_1 < M).$$

(d) Conclude that $\mu = \frac{1}{p_M + p_{M+1} + \cdots}$

(Note: An application of this might be where $Y_1, Y_2, \ldots$ represent successive bids on a certain asset offered for sale. Then $X_n = \max\{Y_1, Y_2, \ldots, Y_n\}$ represents the maximum amount that is bid by stage n. If we suppose that the bid that is accepted is the first bid to exceed some prescribed level M, then μ is the expected time of sale.)

3. Let π_1 and π_2 be two pdfs or pmfs on a set S. For any subset $A \subset A$, $\pi_i(A)$ will denote the probability of being in A under π_i. That is

$$\pi_i(A) = \int_A \pi_i(x)\, dx$$

in the continuous case or

$$\pi_i(A) = \sum_{x \in A} \pi_x$$

in the discrete case.

The *total variation norm distance* between the two distributions is defined as

$$||\pi_1 - \pi_2|| := \sup_{A \subseteq S} |\pi_1(A) - \pi_2(A)|.$$

(Note: This will be the measure of distance we will be using between the limiting distribution of a Markov chain and the distribution of a chain that has run "for a long time" but hasn't reached an equilibrium.)

Show that, in the case of discrete distributions, that

(a) $||\pi_1 - \pi_2|| = \frac{1}{2} \sum\limits_{x \in S} |\pi_1(x) - \pi_2(x)|$

(b) $||\pi_1 - \pi_2|| = 1 - \sum\limits_{x \in S} \min\{\pi_1(i), \pi_2(i)\}$