APPM 5470, Methods of Applied Mathematics: Partial Differential Equations and Integral Equations, Exam 1, 10/9 *Fall 2017*

Answer all the questions and show your work/reasoning. Justify your answers. Partial credit will be given.

1. **Quasi-linear equations:** Consider the Cauchy problem

$$u_x + uu_y = -\frac{1}{2}u, \quad x \neq 0,\ y \in \mathbb{R},$$
$$u(0,y) = f(y), \quad y \in \mathbb{R},$$

where $f \in C^1(\mathbb{R})$.

(a) [15 pts] Does a local solution, i.e., a solution for small $|x|$ and all $y \in \mathbb{R}$, exist? If so, justify your answer. If not, give a counterexample.

(b) [20 pts] Where the solution does exist, determine the solution surface using the method of characteristics.

(c) [15 pts] If $f(y) = y$, does the solution exist globally?

2. **Wave equation:** Consider the following initial-boundary value problem for the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad x > 0, \quad t > 0,\\ u(0,t) &= h(t), \quad t > 0,\\ u(x,0) &= f(x), \quad x > 0,\\ u_t(x,0) &= g(x), \quad x > 0. \end{aligned}$$

(a) [20 pts] Find a solution.

(b) [15 pts] What are the minimal regularity conditions on f, g, and h such that the solution $u(x,t)$ has sufficient regularity to classically satisfy the PDE?

(c) [15 pts] Show that the energy $E(t) = \frac{1}{2}\int_0^\infty (u_t^2 + u_x^2)\,\mathrm{d}x$ is generically not conserved. For what f, g, and h is the energy conserved? You may assume that relevant improper integrals converge.