EXAMPLES OF ENERGY ESTIMATES

Different variations of the 'Energy method' can be used to show that PDEs are 'well posed', to show that discrete approximations are stable, and to establish (global) convergence rates under mesh refinements. The energy approach is very broadly applicable, and can handle many cases which include boundary conditions, variable coefficients, and nonlinearities. However, this flexibility and power comes at a price of often significant technical difficulty.

Example 1: Show that the heat equation

$$u_t = u_{xx}$$

with initial condition $u(x,0) = f(x)$, boundary conditions $u(-1,t) = u(1,t) = 0$ (1)

is 'well posed', i.e. some norm of the solution can be bounded by the initial data:

$$\| u(\cdot, t) \| \le c_1 e^{c_2 t} \| f(\cdot) \| \quad c_1, c_2 > 0 \quad .$$

Choosing to consider the L^2 - norm $\| u \|^2 = \int_{-1}^{1} u^2 dx$, we obtain

$$\frac{\partial}{\partial t} \| u \|^2 = 2 \int_{-1}^{1} u_t \, u \, dx \qquad \text{substitute } u_t = u_{xx}$$

$$= 2 \int_{-1}^{1} u_{xx} \, u \, dx \qquad \text{partial integration; end contributions vanish}$$

$$= -2 \int_{-1}^{1} (u_x)^2 \, dx \le 0 \quad ,$$

i.e.

$$\| u(\cdot, t) \|^2 \le \| f(\cdot) \|^2 \quad .$$

□

Example 2: Show well-posedness when (1) is generalized to

$$u_t = a(x,t) \, u_{xx} + b(x,t) \, u_x + c(x,t) \quad ,$$

where $a(x,t)$ is differentiable and satisfies $a(x,t) \ge a_0 > 0$.

Using the notation

$$(u, v) = \int_{-1}^{1} \bar{u} \cdot v \, dx \quad ,$$

complex conjugation of first argument is often appropriate in cases of u and v complex - not essential issue in the present context

$$\| u \|^2 = (u, u) \quad ,$$

we obtain

$$\frac{\partial}{\partial t} \| u \|^2 \qquad = I + II + III$$

where

$$I \quad = (u, a\, u_{xx}) + (a\, u_{xx}, u) \qquad \text{partial integration}$$

$$= -(u_x, a\, u_x) - (u, a_x u_x) - (a\, u_x, u_x) -$$

use $|(u, a\, v)| \le \| a \|_\infty \| u \| \, \| v \|$ where $\| a \|_\infty = \sup_x |a|$ (a generalization of the Cauchy- Schwarz inequality $|(u, v)| \le \| u \| \, \| v \|$)

$$-(a_x u_x, u) + a\,(\bar{u}\, u_x + \bar{u}_x u)\Big|_{-1}^{1} \qquad \text{boundary terms vanish}$$

$$\le -2\,(u_x, a\, u_x) + 2\, \| a_x \|_\infty \| u \| \, \| u_x \|$$

$$\le -2\, a_0 \| u_x \|^2 + 2\, \| a_x \|_\infty \sqrt{\tfrac{2}{a_0}}\, \| u \| \, \sqrt{\tfrac{a_0}{2}}\, \| u_x \| \qquad \text{use inequality } 2\,a\,b \le a^2 + b^2$$

$$\le -\tfrac{3}{2}\, a_0 \| u_x \|^2 + 2\left(\frac{\| a_x \|_\infty^2}{a_0} \right) \| u \|^2 \quad ,$$

$$II \quad = (u, b\, u_x) + (b\, u_x, u) \qquad \text{use same inequalities as above}$$

$$\le 2\, \| b \|_\infty \| u \| \, \| u_x \|$$

$$\le \left(\frac{\| b \|_\infty^2}{a_0} \right) \| u \|^2 + a_0 \| u_x \|^2 \quad ,$$

$$III \quad = (u, c\, u) + (c\, u, u) \le 2\, \| c \|_\infty \| u \|^2 \quad .$$

Thus

$$\frac{\partial}{\partial t} \| u \|^2 \le -\tfrac{1}{2}\, a_0 \| u_x \|^2 + \alpha \| u \|^2 \qquad \text{where} \quad \alpha = \frac{2\, \| a_x \|_\infty^2 + \| b \|_\infty^2}{a_0} + 2\, \| c \|_\infty$$

$$\le \alpha \| u \|^2 \quad ,$$

and we obtain directly (or by referring to Gronwall's lemma $\phi'(t) \le \alpha\, \phi(t) + g(t) \Rightarrow \phi(t) \le e^{\alpha t}\, \phi(0) + \int_0^t g(s)\, e^{\alpha(t-s)}\, ds$ in its special case of $g(t) \equiv 0$)

$$\| u(\cdot, t) \|^2 \le e^{\alpha t} \| f(\cdot) \|^2 \quad .$$

□

Example 3: Show that the Forward Euler - FD2 scheme

$$\frac{u(x, t+k) - u(x, t)}{k} = \frac{u(x-h, t) - 2\, u(x, t) + u(x+h, t)}{h^2} \qquad (2)$$

applied to (1) is numerically stable for some values of $\lambda = \dfrac{k}{h^2}$.

Max-norm stability:

Only few FD schemes do not allow any growth in the max-norm. In such cases, stability can often be proven very easily. We write the scheme (2) as

$$u(x,t+k) = \lambda\, u(x-h,t) + (1-2\lambda)\, u(x,t) + \lambda\, u(x+h,t).$$

If $0 \le \lambda \le \frac{1}{2}$, the three coefficients $\lambda,\ (1-2\lambda),\ \lambda$ are all non-negative, and add up to 1. Therefore

$$|u(x,t+k)| \ \le\ \max\ \{\, |u(x-h,t)|, |u(x,t)|,\ |u(x+h,t)|\ \}$$

and

$$\|\, u(\cdot,t)\,\|_\infty \le\ \|\, f(\cdot)\,\|_\infty \qquad\qquad (\text{for}\ \ \lambda = \tfrac{k}{h^2} \le \tfrac{1}{2}).$$

L^2-norm stability:

The general procedure we employ here is quite typical for many FD-schemes. We introduce first some convenient notation (somewhat tailored to [-1,1]; $h = \frac{2}{N}$):

$$t_m = m \cdot k\,, \qquad x_i = -1 + ih, \qquad u_i^m = u(x_i, t_m), \qquad \|\, u^m \,\|_{p,s}^2 = h \sum_{i=p}^{s} (u_i^m)^2 \qquad ,$$

$$D_+ u_i \ =\ \tfrac{1}{h}\,[u_{i+1} - u_i]\ \ , \qquad\qquad \text{i.e.}\quad \|\, D_+ u \,\|_{0,N-1}^2 \ =\ \tfrac{1}{h} \sum_{i=0}^{N-1} (u_{i+1} - u_i)^2$$

.

The FD scheme (2) can now be written

$$u_i^{m+1} \,-\, u_i^m \ =\ \tfrac{k}{h^2}\,(\, u_{i-1}^m - 2\, u_i^m + u_{i+1}^m) \quad . \qquad (3)$$

We multiply this by $h\,(u_i^{m+1} + u_i^m)$ and sum over i to obtain

$$\|\, u^{m+1} \,\|_{0,N}^2 - \|\, u^m \,\|_{0,N}^2 \ =$$

$$=\ \tfrac{k}{h} \sum_{i=1}^{N-1} (u_i^{m+1} + u_i^m)(u_{i-1}^m - 2\, u_i^m + u_{i+1}^m)$$

use partial summation; with $u_0 = v_0 = u_N = v_N = 0$
$\sum_{i=1}^{N-1} v_i\,(u_{i-1} - 2\,u_i + u_{i+1}) \ =$
$\quad - \sum_{i=0}^{N-1} (v_{i+1} - v_i)\,(u_{i+1} - u_i)$

$$= -\tfrac{k}{h}\left[\sum_{i=0}^{N-1} (u_{i+1}^{m+1} - u_i^{m+1})(u_{i+1}^m - u_i^m)\right] - \\ -\tfrac{k}{h}\left[\ \sum_{i=0}^{N-1} (u_{i+1}^m - u_i^m)^2\right]$$

use inequality $a\,b \ \le\ \dfrac{a^2 + b^2}{2}$

$$\le \tfrac{k}{2h}\left[\sum_{i=0}^{N-1} (u_{i+1}^{m+1} - u_i^{m+1})^2 + \sum_{i=0}^{N-1} (u_{i+1}^m - u_i^m)^2\right] - \\ - k\, \|\, D_+ u^m \,\|_{0,N-1}^2$$

$$= \frac{k}{2}\,(\,\| D_+u^{m+1}\|^2_{0,N-1} - \| D_+u^m\|^2_{0,N-1}).$$

We introduce now $S^m = \| u^m \|^2_{0.N} - \frac{k}{2}\| D_+u^m\|^2_{0,N-1}$. Because of the inequality above, this quantity satisfies

$$S^{m+1} - S^m = \| u^{m+1}\|^2_{0,N} - \| u^m\|^2_{0,N} - \frac{k}{2}\,(\,\| D_+u^{m+1}\|^2_{0,N-1} - \| D_+u^m\|^2_{0,N-1}) \le 0 \quad ,$$

i.e. it cannot grow with *m*. Stability of the numerical scheme (2) will follow if we can show that this bound on S^m implies a bound on $\| u^m\|^2_{0,N}$ (note that no condition on $\lambda = \frac{k}{h^2}$ has entered yet - it must come in now). We get

$$S^m = \| u^m\|^2_{0,N} - \frac{\lambda h}{2}\sum_{i=0}^{N-1}(u^m_{i+1} - u^m_i)^2 \qquad \text{use } (a - b)^2 = a^2\text{-}2ab + b^2 \le 2(a^2 + b^2)$$

$$\ge \| u^m\|^2_{0,N} - 2\lambda h\sum_{i=0}^{N}(u^m_i)^2 \qquad \text{assume now that } \lambda \le \tfrac{1}{2}(1-\varepsilon) \text{ for some } \varepsilon > 0.$$

$$= \| u^m\|^2_{0,N} - (1-\varepsilon)\| u^m\|^2_{0,N} = \varepsilon\,\| u^m\|^2_{0,N} \quad .$$

Therefore,

$$\| u^m\|^2_{0,N} \le \tfrac{1}{\varepsilon}S^m \le \tfrac{1}{\varepsilon}S^0 \le \tfrac{1}{\varepsilon}\,\| f\|^2_{0,N}$$

holds if $\lambda \le \frac{1}{2}(1-\varepsilon)$. Note that this argument gave a sufficient, but not necessary condition for stability - it failed to establish stability when $\lambda = \frac{1}{2}$.

The case $\lambda = \frac{1}{2}$ can be taken care of separately. Equation (3) becomes in this case

$$u^{m+1}_i = \tfrac{1}{2}\,(u^m_{i-1} + u^m_{i+1}) \quad .$$

Squaring both sides give

$$(u^{m+1}_i)^2 = \tfrac{1}{4}\,(u^m_{i-1} + u^m_{i+1})^2 \qquad \text{use inequality } (a+b)^2 \le 2(a^2+b^2)$$

$$\le \tfrac{1}{2}\,((u^m_{i-1})^2 + (u^m_{i+1})^2) \quad ; \qquad \text{sum over } i$$

$$\| u^{m+1}\|^2_{0,N} \le \tfrac{1}{2}\| u^m\|^2_{0,N} + \tfrac{1}{2}\| u^m\|^2_{0,N} = \| u^m\|^2_{0,N} \quad . \qquad \square$$

Example 4: Show that the equation

$$\frac{\partial u}{\partial t} + \frac{\theta}{2}\frac{\partial u^2}{\partial x} + (1-\theta)\, u\, \frac{\partial u}{\partial x} = 0$$

for $\theta = \frac{2}{3}$ is stable when approximated in space by $\frac{\partial}{\partial x} = D_0$ and left continuous in time.

We first note that the case $\theta = \frac{2}{3}$ is the only one for which the period-3 pattern $\ldots 0 \quad \varepsilon(t) \quad -\varepsilon(t) \quad 0 \quad \varepsilon(t) \quad -\varepsilon(t) \quad 0 \ldots$ does not go to infinity at a finite time, according to the relation

$$\frac{d\,\varepsilon(t)}{dt} = \frac{c_p\,(1-3\theta/2)}{h}\,\varepsilon^2(t)$$

where c_p is a constant which depends on the space approximation; $c_p = \frac{1}{2}$ for $\frac{\partial}{\partial x} = D_0$.

Considering a periodic- or infinite-domain problem, we get

$$\frac{d}{dt}\sum_i u_i^2 = 2\sum_i u_i \frac{d u_i}{dt} = -\tfrac{2}{3}\sum_i u_i\, D_0 u_i^2 - \tfrac{2}{3}\sum_i u_i^2 D_0 u_i = 0$$

since (by reordering of the sum)

$$\sum_i u_i\, D_0 u_i^2 = \tfrac{1}{2h}\sum_i u_i\,(u_{i+1}^2 - u_{i-1}^2) = -\tfrac{1}{2h}\sum_i u_i^2\,(u_{i+1} - u_{i-1}) = -\sum_i u_i^2\, D_0 u_i\ .$$

□