APPM 2460
EIGENSTUFF IN MATLAB

1. Introduction

Given a matrix A, we define the eigenvalues and eigenvectors of A by the following relation:

$$Av = \lambda v$$

In words, we need A times the eigenvector to return the eigenvector multiplied by its associated eigenvalue. Note that v is not unique. That is, we can multiply it by any constant and it is still an eigenvector.

2. Finding Eigenvalues

To find the eigenvalues of A, consider

$$Av = \lambda v \iff (A - \lambda I)v = 0$$

Recall that this homogeneous system has the unique solution if and only if $|A - \lambda I| \neq 0$. In this case, the only solution is $v = 0$. To get something interesting we seek

$$|A - \lambda I| = 0$$

This gives a polynomial in terms of λ, which we call the characteristic polynomial. The roots of this polynomial are the eigenvalues of A. Lets find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

The characteristic equation for A is

$$p(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

and we would like to find its roots. The first step is to set up a function for this characteristic polynomial. We can write it as an anonymous function:

```
char_eq = @(x) x^3 - 6*x^2 + 11*x - 6;
```

To find the roots of this equation, we use the `fzero` command. Type `help fzero` to see how to use it. In our case, we can type:

```
eigval1=fzero(@(x) ch_eq(x),4)
```

where the `@(x)` tells `fzero` what variable our function is in terms of and 4 is our guess as to where the root is. To get the other eigenvalues, use the following snippets:

```
eigval2=fzero(@(x) ch_eq(x),2.5)
eigval3=fzero(@(x) ch_eq(x),0.5)
```

3. Finding Eigenvectors

Each one of our eigenvalues has an eigenvector associated with it. To find the eigenvector, we solve the system $|A - \lambda I| = 0$. Type

```
A=[1,2,-1;1,0,1;4,-4,5]
B=A-eigval1*eye(3);
rref(B)
```

We see from the rref of B that the eigenvector associated with eigenvalue, `eigval1` is $\mathbf{v_1} = c[-1\ 1\ 4]^T$, where c is an arbitrary constant. We can either choose a c that makes the length of v_1 be 1, or we can choose a c that makes the entries of v_1 not have fractions.

4. Finding Eigenvalues Quickly

That was fun, and we recalled how to use the command, `fzero`. However, it took a while and we had to find the characteristic equation by hand. A much easier way of finding the eigenvalues of a matrix is the `eig` command. Try typing

```
eigenvalues = eig(A)
```

5. Finding Eigenvectors Quickly

To find the eigenvalues and eigenvectors all at once, type

```
[V D] = eig(A)
```

V is a matrix, the columns of which are the eigenvectors associated with D, the diagonal matrix where each element of the diagonal is an eigenvalue of A. The eigenvector in column i of V is associated with the eigenvalue in column i of D. The eigenvectors we see have been normalized to have length 1, potentially making them ugly. To have Matlab not do this, try

```
[V D] = eig(A,'nobalance')
```

6. Homework #8

If a matrix A has dimension $n \times n$ and has n linearly independent eigenvectors, it is diagonalizable. This means there exists a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix, and the diagonal is made up of the eigenvalues of A. P is constructed by taking the eigenvectors of A and using them as the columns of P. Your task is to write a program (function) that does the following

- Finds the eigenvectors of an input matrix A
- Checks if the eigenvectors are linearly independent (think determinant)
  - if they are not linearly depended, exit the program & display error
- Displays P, P^{-1} and D (if possible)
- Shows that $PDP^{-1} = A$

Show that your program works with a 3×3 matrix A.

Interesting Fact: Even if a matrix is not diagonalizable, we can get pretty close. Every matrix has something called a Jordan canonical form. For a diagonalizable matrix, this is just the diagonal form, but if we have insufficient eigenvectors, there will be the number 1 in the upper diagonal above the deficient eigenvalues on the diagonal. We construct P with something called the generalized eigenvectors. For more information, type `help jordan`. This is an extremely important theorem of linear algebra!