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Dynamics on networks with higher-order interactions

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INTRODUCTION

The study of dynamical processes in networked systems is one of the central problems in complexity science.[1,2] Even simple dynamical systems, when connected with each other, can produce complex collective behavior. Examples include synchronization in coupled oscillator networks and spreading of opinions, information, and disease in social networks. Most studies of dynamical systems on networks assume that the interactions between these systems can be described as a collection of pairwise interactions, described by a network made of nodes and links.[3] This framework has been very fruitful, yielding important insights into the effect of the network structure on epidemic spreading,[4,5] synchronization,[5] percolation,[3] and many other dynamical phenomena.

While the assumption that interactions between dynamical systems occur in pairs is often valid, there are many situations where interactions between more than two systems occur simultaneously in a way that cannot be reduced to multiple pair interactions. Examples include social dynamics,[6,7] interactions in ecological systems,[8–10] neuroscience,[11] and coupled oscillators.[12,13] Accordingly, there has been a focused effort by the complex systems community to ascertain how these types of interactions, called "higher-order interactions," modify the collective behavior of networked dynamical systems. For example, when higher-order interactions are included in the prototypical susceptible-infected-susceptible (SIS) model of epidemic spreading, the dynamics are fundamentally modified, admitting bistable solutions, explosive transitions, and hysteresis.[14,15] Similarly, with higher-order interactions, the Kuramoto model of phase oscillator synchronization shows a much richer dynamical landscape than without them.[16,17] The effects of higher-order interactions on other coupled dynamical systems are summarized, for example, in Refs. 18–21. We emphasize, however, that while the role of higher-order interactions has been fully embraced by the community only recently, there have been various pioneering works extending back many years (e.g., Refs. 22–26).

As this research direction has matured, different thrusts have emerged. One thrust is to find how the presence of higher-order interactions can modify the dynamics of systems that have been well-studied in the context of pairwise interactions, such as the Kuramoto model of synchronization, epidemic models, or diffusion models. Another direction is to study novel dynamical systems defined in terms of higher-order interactions (e.g., Refs. 27 and 28). A third thrust is to formalize the theoretical bases of dynamics with higher-order interactions and to develop generative models and quantitative descriptions of the structure of higher-order interactions. The "Dynamics on Networks with Higher-Order Interactions" Focus Issue in *Chaos* presents key developments in all these areas and gives an overview of this rapidly evolving field.

THE "DYNAMICS ON NETWORKS WITH HIGHER-ORDER INTERACTIONS" FOCUS ISSUE

Below we summarize the contributions of the papers in the "Dynamics on Networks with Higher-Order Interactions" Focus Issue in *Chaos*. For convenience, we have tried to organize the papers in distinct categories, even though some papers do not fall neatly into only one.

Synchronization

Complex networks are one of the fundamental topics in current research to describe many real phenomena in biology, physics, and engineering sciences. When studying complex dynamical networks where the nodes represent dynamical systems, one of the most significant phenomena is the emergence of collective states like synchronization. Synchronization is the coherent dynamics that emerge in coupled systems. Previously, different types of synchronization states were observed using different kinds of network topologies that can be static[29] or time-varying[30] in nature. In recent studies, it has been shown that including higher-order interactions is necessary to model many real-world phenomena.[18–20] In the last few decades, many studies on synchronization of identical systems have been implemented using pairwise interaction networks and analytically studied using the master stability function (MSF) approach.[31] Thus, one approach to analytically study synchronization in systems with higher-order interactions involves developing an extended version of the MSF.[32] Moreover, the study of synchronization in time-varying higher-order networks[33] is also at an early stage and can be tackled similarly. We refer the reader to review articles,[19,20] where many different properties of higher-order interactions are discussed together with collective phenomena, including synchronization, chimera states,[34] contagion dynamics, etc.

The emergence of synchronization of coupled phase oscillators on hypergraphs is an interesting topic. In this connection, Adhikari *et al.*[35] have developed a general formalism to study synchronization of phase oscillators on hypergraphs. To illustrate it, they generated hypergraphs through two different mechanisms: the former generates a random hypergraph where both pairwise and higher-order interactions are constructed randomly, while the other one generates a hypergraph with correlated links and triangles, and the number of pairwise and triadic interactions is correlated to each other. The authors show that for both types of hypergraphs, an abrupt transition to synchrony with associated hysteresis emerges under sufficiently strong triadic coupling. For the correlated hypergraph, the onset of abrupt synchronization and bistability depends on the moments of the degree distribution. Furthermore, the triadic coupling only affects the emergence of bistability but not the commencement of synchrony. By reducing the system of differential equations in terms of the structural characteristics of the hypergraph, they derive analytically the prerequisites for the onset of abrupt synchronization and bistability.

The construction and emerging synchronization phenomena in multiplex hypergraphs is another interesting topic. In multiplex networks, two types of interactions are present, namely, intralayer interaction within network layers and interlayer interactions between layers. In Ref. 36 the authors construct multiplex hypergraph networks in which intralayer interactions are considered to be higher-order, constructed by hypergraphs, and the interlayer connections are pairwise interaction between nodes of different layers. As in previous studies of synchronization in multiplex network structures, only pairwise interactions between the units in the layers are considered. In this network, two types of synchronization phenomena in the multiplex hypergraph emerge: intralayer and interlayer synchronization. Compared to the pairwise multiplex networks, where the intralayer connections are described by graphs, Anwar and Ghosh[36] unveil a significant improvement in intralayer synchrony for multiplex hypergraphs. Nevertheless, the underlying behavior of interlayer synchronization remains almost the same in both scenarios. Furthermore, the enhancement in intralayer synchrony is analytically supported by calculating the spectral gap of Laplacian matrices corresponding to the multiplex hypergraph and pairwise multiplex network. They also illustrated that the interlayer synchrony in multiplex hypergraphs is more robust to random removal of interlayer links when compared with pairwise multiplex networks.

In another study, Parastesh *et al.*[37] investigate the effect of higher-order interactions, particularly triadic interactions, on the emergence of complete synchronization in globally coupled Hindmarsh–Rose neurons. Assuming dyadic interactions to be mediated through electrical synapses, the occurrence of synchrony is also explored for two different instances of three-body interactions mediated through linear diffusive and nonlinear chemical synaptic couplings. Their results show a sufficient enhancement in synchrony for both cases as compared to the solely pairwise scenario. Furthermore, to quantify the enhancement in synchrony due to the inclusion of three-body interactions, the authors introduced a synchronization cost measure based on coupling strengths and interactions. It shows that in both scenarios, the cost of synchronization is decreased when compared with the pairwise situation. However, the cost of synchronization is higher for nonlinear synaptic coupling than for linear diffusive coupling.

In another work, Skardal *et al.*[38] have investigated the combined effect of higher-order interactions and community structure in ensembles of phase oscillators. The combination of these two induces several novel states that are unsupported by either of them alone. In addition to finding expected states such as when both communities are synchronized, both are desynchronized, and one is synchronized while the other is in a desynchronized state, the authors also find two new states: an antiphase synchronized state where both communities are in the synchronized state but with opposite phases; and a skew-phase synchronized state where both communities are synchronized but oscillate with a nonzero phase difference that depends on the strength of intercommunity coupling. The authors support their observations by deriving the low dimensional dynamics and analyzing the bifurcation of the system using perturbation theory and stability analysis.

In Ref. 39, the authors investigate a higher-order interaction model on the sphere. Specifically, a system with N interaction particles is considered on the unit sphere in d-dimensional space. Then the Kuramoto model is written as a gradient flow of a suitably defined potential. In this system, the synchronization state is also controlled by higher-order interactions. Multistability is a general phenomenon in higher-order interaction networks, but in the proposed model on the sphere, the multistability can be controlled by deleting the signature factor in the connectivity coefficient.

Finally, Ziegler *et al.*[40] develop and study a model for consensus over simplicial complexes based on the Hodge Laplacian matrix (which generalizes the graph Laplacian). Linear models for consensus dynamics are popular models for synchronization in the context of collective decision making and decentralized machine learning and also arise for nonlinear systems if one examines linearized, perturbative states near a synchronization manifold. Their work introduces a balanced Hodge Laplacian in which the strength of higher- and lower-order interactions can be tuned and optimized to maximize the convergence rate. Their work also reveals that the harmonic subspace of a Hodge Laplacian, which is determined by the homology of a simplicial complex, acts as a low-dimensional attractor for the collective dynamics.

Contagion, spreading, and diffusion processes

Understanding how things spread across networks is paramount to numerous endeavors including the study of epidemics, social contagions, cascading failures and blackouts, neuronal avalanches, and much more.[41,42] As such, there has been considerable work to investigate spreading processes in generalized settings including multilayer and temporal networks. Developing models and theory for spreading processes over hypergraphs and simplicial complexes is also a rapidly growing pursuit, with recent extensions including the theory for epidemics[43] and random walks.[44] Notably, the study of (conservative) random walks and diffusion over hypergraphs and simplicial complexes has for some time been a topic of interest in mathematics and computer science;[45–48] however, until recently, there has been comparatively little work on higher-order models for spreading processes that are non-conservative. Below, we survey several recent works that study such dynamics over hypergraphs and simplicial complexes.

In Ref. 49, Higham and De Kergorlay extend the susceptible-infected-susceptible (SIS) epidemic model to the setting of time-varying hypergraphs. First, they study a discrete-time network model defined as a sequence of graphs that are i.i.d. random samples from a Gilbert random-graph model. Next, they propose an extension of the Gilbert model to hypergraphs using it to similarly define time-varying hypergraphs. For both models, the authors develop a mean-field theory to describe SIS epidemic spreading over time-varying graphs/hypergraphs and investigate the epidemic threshold that can determine the long-time extinction of a spreading process. Their main finding is that the spectral-based criterion for determining epidemic extinction can be expressed in terms of a static, expected affinity matrix (or expected clique expansion in the hypergraph case).

It is also important to study spreading dynamics in the context of competing contagions, which can model, e.g., the competition between vaccination, misinformation, and epidemic spreading. In Ref. 50, Li *et al.* extend the study of the susceptible-infected-recovered (SIR) epidemic model to the context of competing epidemics over simplicial complexes. They develop theory using the microscopic Markov chain approach and study various dynamical properties that arise including an epidemic-free state, co-existing epidemics, and one epidemic dominating over the other. The authors study the impact of competition and higher-order interactions on these phenomena as well as the epidemics' growth rates.

Complex contagions complement the study of epidemics by considering when transmission events require multiple exposures and are often modeled using a threshold mechanism. In Ref. 51, Xu *et al.* extend the study of threshold-based contagion models to the setting of hypergraphs. They develop generating functions and mean-field theory to characterize cascades over synthetic and empirical hypergraphs and they conduct experiments to study the effects of heterogeneity for both the thresholds and that of the degrees and hyper-degrees. Motivated by applications in which cascades have a negative connotation (e.g., cascade failures), the authors study how higher-order interactions and heterogeneity can influence a systems robustness against large-scale cascades.

Ghasemi and Kantz[52] also study cascades over hypergraphs but focus on the context of cascading transmission-line failures for power grids. Complementing the development of bifurcation theory for cascade models, the authors instead focus on the "inverse" problem in which one seeks to develop data-driven models for line failures based on time series data summarizing historic line failures. They identify pairwise and higher-order (indirect) dependencies among transmission lines by combining a weighted l1-regularization approach with pairwise maximum entropy. The approach involves predicting dependencies by maximizing the log-likelihood of a line's state given the states of its neighbors. Informed by their data-reconstructed model, they investigate cascades using a Glauber model and use simulations to predict the cascade size distribution, infer co-susceptible line groups, and compare the results against the data.

In Ref. 53, Klimm also adopts a data-driven perspective for studying cascade dynamics by applying topological data analysis to time series data encoded in cascade maps. This approach involves identifying and studying dynamical bifurcations by examining topological changes in related data, and, in particular, changes for topological features that can be identified and examined using the toolbox of persistent homology. Extending prior work in this area, this work explores how this approach can be improved by truncating time-series data, which is supported with experiments for both synthetic and empirical network datasets. As a concrete application, Klimm highlights the utility of the cascade-based, manifold-learning technique to uncover a differentiation trajectory for single-cell transcriptomics data of mouse oocytes.

Hypergraph games and competition dynamics

The study of multiplayer games and competition dynamics on networks is relevant in fields such as economics, social science, and evolutionary biology. In the "Dynamics on Networks with Higher-Order Interactions" Focus Issue, various papers consider the effect of multiplayer interactions on various aspects of game and competition dynamics.

The mean-field analysis of a large network and hypergraph dynamical systems is often carried out without a solid theoretical foundation. Addressing this, the paper by Cui *et al.*[54] provides a rigorous framework for the study of multiplayer games on large, dense hypergraphs by combining the tools of mean-field games with hypergraphons, a limiting description of large hypergraphs. The authors are able to rigorously describe and analyze the resulting large systems of non-linear, weakly interacting dynamical agents. The authors illustrate their techniques with a rumor spreading example and an epidemics control problem.

In many cases, the number of players interacting in a game is not always the same. Kontorovsky *et al.*[55] study this situation by considering a game where the number of players participating in each interaction is randomly chosen. Generalizing the concept of an evolutionary stable strategy (ESS) to this case, the authors show that the existence of such a strategy depends on the probability that the interactions are pairwise. Furthermore, the authors propose and study an agent-based model where agents interact in pairs ("duels") or triples ("truels") and find good agreement with their mean-field predictions.

The persistence of biodiversity in the presence of competitive species interactions is an important problem in ecology. The possibility that higher-order interactions contribute to preserve biodiversity is explored in the paper by Chatterjee *et al.*[56] In this paper, the authors study a simple model for the dynamics of species densities that includes higher-order interactions. They find that their model leads to species co-existence and diversity. In addition, the authors study how perturbations to the interaction strengths between species can eventually lead to various effects in the density of all the species in the system. Interestingly, the authors find that small perturbations can lead to the formation of synchronized clusters.

Finally, the article by Schlager *et al.*[57] studies how the stability of equilibrium points in evolutionary games is affected by multi-player interactions and network adaptation. Using the Snowdrift game with rational adaptation rules as an example, the authors show by means of direct simulations and analytical calculations that the stable equilibrium of the game is remarkably robust. In particular, the stability of these fixed points is not affected by the introduction of higher-order interactions. The results of this paper suggest that despite the existence of some prominent and interesting examples, higher-order interactions do not always fundamentally change the dynamics of networked systems.

Effects of hypergraph structure on dynamical systems

One of the central goals of modern network science has been to determine how the structure of the interaction network affects the collective dynamics of networked systems. Two articles in the "Dynamics on Networks with Higher-Order Interactions" Focus Issue make contributions to this task for networks with higher-order interactions.

The article by Nijholt *et al.*[58] considers a general model for nonlinear dynamical systems defined on a simplicial complex. The authors show that the importance of algebraic structures defined on simplicial complexes survives when considering nonlinear dynamical systems. They then identify the symmetries of the dynamics induced by the simplicial complex structure, study the effect of the simplicial complex orientation on the dynamics, and relate the symmetries to invariant subspaces of the dynamics. The authors, thus, provide a comprehensive study of how the structure of a simplicial complex affects dynamical processes occurring on it.

For pairwise networks, the dominant eigenvalue of the network adjacency matrix is determinant in many network dynamical processes. The paper by Landry *et al.*[59] studies the analogous version of this eigenvalue, which they call the "expansion eigenvalue," for hypergraphs. First, the authors give an approximation to the expansion eigenvalue for random hypergraphs in terms of the hyperdegree distribution, and then they use a perturbative expansion to give an approximation for correlated hypergraphs. The authors then show how their approximation can be used to modify the connections of the hypergraph to alter the behavior of dynamical processes.

CONCLUSIONS

The articles in the "Dynamics on Networks with Higher-Order Interactions" Focus Issue highlight different important directions in the study of dynamics in networks with higher-order interactions, an active and rapidly developing field. We hope that this collection stimulates additional research in the area. Finally, we would like to thank all the authors and referees who contributed to this Focus Issue.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Z. Gao: Conceptualization (equal). **D. Ghosh:** Conceptualization (equal). **H. Harrington:** Conceptualization (equal). **J. G. Restrepo:** Conceptualization (equal). **D. Taylor:** Conceptualization (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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