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Regularization of Vector Fields by Surgery*

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The regularization of the planar 2-body problem on surfaces of constant energy was first discussed by Levi–Civita. The 2-body problem is regularized in Section 2 below using surgery. This new method gives a new way of looking at the classical result and makes apparent the geometric reasons for its success. The regularization is achieved for all energy surfaces at once, and, therefore, makes possible regularization of the 3-body problem on surfaces of nonzero angular momentum. This result will be treated in a subsequent paper.

An unsolved problem is the problem of regularizing the singularity of tripple collision in the 3-body problem. This singularity is contained in an isolating block (for definition, see 1.4) and the geometry of the isolating block sheds some light on the geometry of orbits near tripple collision. This looks like a promising topic for further research.

The method of regularization of vector fields by surgery will be discussed in detail in Section 1. Roughly, the idea is to excise a neighborhood of the singularity from the manifold on which the vector field is defined and then to identify appropriate points on the boundary of the region. Thus, for example, if we consider the differential equation on R^2 defined by $\dot{x} = (x^2 + y^2)^{-1}$, $\dot{y} = 0$. We can remove the singularity $x = y = 0$ by cutting a disk centered at the origin from the plane and identifying points on the boundary of this disk which have the same y component. The identification pairs points which lie on the same integral curves of the vector field (except for the pair of points on the x-axis).

1. Isolating Blocks and Surgery

Let M be a C^∞ manifold of dimension d, let S be a closed subset of M, and let V be a C^1 vector field defined on $M - S$.

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Definition 1.1. An *integral curve* of V through $p \in M - S$ is a C^1 function $\gamma : (a, b) \to M - S$ such that $\gamma(0) = p$ and the tangent vector to γ at the point $\gamma(s)$ is equal to $V(\gamma(s))$ for each $s \in (a, b)$. Since V is C^1, integral curves of V through p exist and agree on their common domain of definition.

Definition 1.2 (Notation). Let $p \in M - S$. If γ is an integral curve of V through p then $p \cdot t$ will denote the point $\gamma(t)$ for each t in the domain of definition of γ. If t does not belong to the domain of definition of any integral curve of V through p, then $p \cdot t$ will not be defined. Also for $A \subset M - S$ and $T \subset R^1$, if $p \cdot t$ is defined for each $p \in A$ and $t \in T$, we define $A \cdot T = \{p \cdot t \mid p \in A, t \in T\}$. We call $p \cdot t$ the *action on* $M - S$ *induced by* V.

Let N be a smooth submanifold with boundary of M with $\dim N = \dim M$ and let n equal the boundary of N. Suppose $n \cap S = \phi$.

Definition 1.3.

$$n^+ = \{p \in n: \ \text{for some } t > 0 \quad p \cdot (-t, 0) \cap N = \phi\}$$

$$n^- = \{p \in n: \ \text{for some } t > 0 \quad p \cdot (0, t) \cap N = \phi\}$$

$$\tau = \{p \in n: \ V \text{ is tangent to } n \text{ at } p\}.$$

In general, n^+, n^- and τ may be variously related but their union must always be n. n^+, n^- and τ might be called the ingress, egress and tangency points of n, respectively.

Definition 1.4. With the notation of 1.3, N is an *isolating block* (for V) if $n^+ \cap n^- = \tau$, and if τ is a smooth submanifold of n with codimension one and (as a consequence) n^+ and n^- are submanifolds of n with common boundary τ.

Definition 1.5. With the notation of 1.4 we define

$$A^+ = \{p \in N: p \cdot t \in N \text{ for each } t \geqslant 0 \text{ for which } p \cdot t \text{ is defined}\}$$

$$A^- = \{p \in N: p \cdot t \in N \text{ for each } t \leqslant 0 \text{ for which } p \cdot t \text{ is defined}\}$$

$$I = A^+ \cap A^-$$

$$a^+ = A^+ \cap n^+, \qquad a^- = A^- \cap n^-.$$

Definition 1.6. $\pi^+ : n^+ - a^+ \to n^- - a^-$ is defined by $\pi^+(p) = p \cdot t^+$ where $t^+ = \sup\{t \geqslant 0 : p \cdot t \subset N\}$. It is shown in [1] that π^+ is a homeomorphism of $n^+ - a^+$ onto $n^- - a^-$. π^+ is differentiable in the present case since V is of class C^1 and n is a smooth submanifold of M. Thus π^+ is a diffeomorphism of $n^+ - a^+$ onto $n^- - a^-$. A comprehensive treatment of isolating blocks (and isolated invariant sets) is given in [1].

DEFINITION 1.7. Suppose that N is an isolating block for V, and suppose $S \subset N$. With the notation of 1.5 and 1.6, suppose that π^+ admits an extension as a diffeomorphism from n^+ to n^-. Then the singularity S is said to be *regularizable*.

DEFINITION 1.8. If S is regularizable, with the notation of 1.7, define a manifold $M' = M - \operatorname{int} N/\pi^+$ called the *regularized manifold* as follows: Define an equivalence relation $\sim$ on $M - \operatorname{int} N$ by $x \sim y$, if $x = y$ or if $x = \pi^+(y)$, or if $y = \pi^+(x)$, and let M' be the set of equivalence classes of points of $M - \operatorname{int} N$. Let $\rho : M - \operatorname{int} N \to M'$ be the natural projection of $M - \operatorname{int} N$ onto M' defined by letting $\rho(x)$ be the equivalence class of x. M' is topologized by defining a set $U \subset M'$ to be open if and only if $\rho^{-1}(U)$ is open in $M - \operatorname{int} N$. Then ρ restricted to $M - N$ is a homeomorphism onto its image. We omit the argument that M' is a manifold. Since $\rho : M - N \to M'$ is a homeomorphism onto its image we will identify $M - N$ and $M' - \rho(n)$. Thus, it is easy to see that at least $M' - \rho(n)$ can be given the structure of a differentiable manifold. In fact, M' can be given the structure of a differentiable manifold although care must be taken in defining coordinate charts containing points of $\rho(\tau)$.

DEFINITION 1.9. For $p \in M'$ and $t \in R^1$, we define $p * t \in M'$ as follows:

(1) Suppose $p \in M - N$. If $p \cdot s \in M - N$ for each s between 0 and t, then define $p * t = \rho(p \cdot t)$.

(2) Suppose $p \in \rho(n)$, say, $\rho^{-1}(p) = \{x, y\}$, where $x \in n^+$ and $y = \pi^+(x)$. If $t \geqslant 0$ and $y \cdot (0, t) \subset M - N$, define $p * t = \rho(y \cdot t)$. If $t \leqslant 0$ and $x \cdot (t, 0) \subset M - N$, define $p * t = \rho(x \cdot t)$.

(3) Extend the action $p * t$ by requiring that $p * (t_1 + t_2) = (p * t_1) * t_2$.

Remark. Suppose the action $(x, t) \to (x \cdot t)$ defined on $M - S$ has the property that if $x \cdot t_0$ is undefined, and if $x \cdot t$ is defined for t between 0 and t_0, then $x \cdot t$ approaches S as $t \to t_0$. Then the action $p * t$ defined on M' is a flow. This means that $p * t$ is defined for all $p \in M'$ and $t \in R^1$ and $* : M' \times R^1 \to M'$ is a one-parameter family of homeomorphisms of M'.

2. REGULARIZATION OF THE 2-BODY PROBLEM

The problem of describing the motion of two mass particles under the influence of their mutual gravitational attraction is known as the 2-body problem. By choosing a coordinate system with origin at the center of mass of the two bodies the problem is reduced to the study of the motion of one mass point only. The motion can be described by a Hamiltonian system of differential equations

$$\dot{x} = \frac{\partial H}{\partial y}(x, y), \quad \dot{y} = -\frac{\partial}{\partial x} H(x, y), \tag{2.1}$$

where $S = 0 \times R^3$ and $H : R^6 - S \to R^1$ is defined by $H(x, y) = \frac{1}{2} | y |^2 - | x |^{-1}$. A complete description of the solutions to this system of equations has been obtained classically using the three known integrals of motion, viz., the total energy $H(x, y)$, the angular momentum $W(x, y) = (x) \times (y)$ and the third integral $E(x, y) = (y) \times W(x, y) - x | x |^{-1}$.

The planar 2-body problem can be regularized on a surface of constant energy $H(x, y) = h < 0$ by considering x and y as complex variables and introducing new coordinates (ξ, η) by the transformation $(\xi, \eta) \to (\xi^2/2, \eta/\bar{\xi})$ and by introducing, as the new time variable, $s = \int dt/| x |$. This is the classical result of Levi–Civita.

Regularization of the 2-body problem by surgery is carried out below. The first step is to construct an isolating block $N \subset R^6$ for Eq. (2.1) which contains the singularity S.

DEFINITION 2.2. $f : R^6 - S \to R^1$ is defined by $f(x, y) = \frac{1}{2}(x, x) - \alpha(H(x, y))$ where $\alpha : R^1 \to (0, \infty)$ is a smooth function to be determined below. Let $\dot{f}(x, y) = d/dt\, f(x(t), y(t))|_{t=0}$ where $x(t)$, $y(t)$ is the solution to Eq. (2.1) satisfying initial condition $(x(0), y(0)) = (x, y)$. Let

$$\ddot{f}(x, y) = \frac{d^2}{dt^2} f(x(t), y(t)) \Big|_{t=0}.$$

$\dot{f}(x, y)$ and $\ddot{f}(x, y)$ can be computed without solving (2.1) by using the chain rule. Thus

$$\dot{f}(x, y) = (x, y) \qquad \text{and} \qquad \ddot{f}(x, y) = 2H(x, y) + | x |^{-1}.$$

The function α is chosen to be a sufficiently close smooth approximation to the function

$$\alpha_1(h) = \begin{cases} \frac{1}{2} & \text{if } h \geqslant 0 \\ \frac{1}{2}(1 - 2h)^{-2} & \text{if } h \leqslant 0, \end{cases}$$

so that when $f(x, y) = 0$ and $\dot{f}(x, y) = 0$, then

$$\ddot{f}(x, y) = 2H(x, y) + [2\alpha(H(x, y))]^{-1/2} > 0.$$

This choice of α is motivated by the following definition.

DEFINITION 2.3. $N = \{(x, y) \in R^6 : f(x, y) \leqslant 0\} \cup S$. Define sets n, n^+, n^-, τ, a^+, a^- and map $\pi^+ : n^+ - a^+ \to n^- - a^-$ as in Definitions 1.3 to 1.6.

We will show that N is an isolating block for (2.1), making use of the fact that when $f = 0$ and $\dot{f} = 0$ we have $\ddot{f} > 0$.

DEFINITION 2.4.

$$\begin{aligned}
N_1 &= R^1 \times (-\infty, 0] \times S^2 \times S^2 \subset R^1 \times R^1 \times R^3 \times R^3 \\
n_1 &= R^1 \times 0 \times S^2 \times S^2 \\
n_1^+ &= R^1 \times 0 \times \{(u, v) \in S^2 \times S^2 : (u, v) \leqslant 0\} \\
n_1^- &= R^1 \times 0 \times \{(u, v) \in S^2 \times S^2 : (u, v) \geqslant 0\} \\
\tau_1 &= R^1 \times 0 \times \{(u, v) \in S^2 \times S^2 : (u, v) = 0\} \cdot \\
a_1^+ &= R^1 \times 0 \times \{(u, v) \in S^2 \times S^2 : (u, v) = -1\} \\
a_1^- &= R^1 \times 0 \times \{(u, v) \in S^2 \times S^2 : (u, v) = 1\}.
\end{aligned}$$

LEMMA 2.5. *N is an isolating block for Eq. (2.1). Furthermore, there exists a diffeomorphism $\Gamma : N \to N_1$ taking n to n_1, n^+ to n_1^+, n^- to n_1^-, τ to τ_1, a^+ to a_1^+ and a^- to a_1^-.*

Proof. N is a submanifold with boundary of R^6, since grad f does not vanish on n. Define $\Gamma(x, y) = (H(x, y), f(x, y), x/| x |, y/| y |)$. Γ is clearly a diffeomorphism. To see that Γ takes n to n_1 etc. we first characterize the sets n, n^+, n^-, τ, a^+ and a^-. Clearly,

$$\begin{aligned}
n &= \{(x, y) : f(x, y) = 0\} \\
n^+ &= \{(x, y) : f(x, y) = 0, \dot{f}(x, y) \leqslant 0\} \\
n^- &= \{(x, y) : f(x, y) = 0, \dot{f}(x, y) \geqslant 0\} \\
\tau &= \{(x, y) : f(x, y) = 0, \dot{f}(x, y) = 0\}.
\end{aligned}$$

$\tau \subset n^+$ and $\tau \subset n^-$ since for $(x, y) \in \tau$ we have $\ddot{f}(x, y) > 0$. Thus Γ takes n to n_1, n^+ to n_1^+, n^- to n_1^- and τ to τ_1. It follows that τ is a smooth submanifold of n since Γ is a diffeomorphism and τ_1 is a smooth submanifold of n_1. Hence N is an isolating block for Eq. (2.1). a^+ and a^- are characterized by the equalities

$$\begin{aligned}
a^+ &= \{(x, y) \in n : (x) \times (y) = 0 \text{ and } (x, y) < 0\} \\
a^- &= \{(x, y) \in n : (x) \times (y) = 0 \text{ and } (x, y) > 0\},
\end{aligned}$$

and it is easy to verify that $\Gamma(a^+) = a_1^+$ and $\Gamma(a^-) = a_1^-$. This completes the proof.

In order to show that the singularity S is regularizable we must prove that the map $\pi^+ : n^+ - a^+ \to n^- - a^-$ extends to a diffeomorphism of n^+ onto n^-. Suppose $(x, y) \in n^+ - a^+$ and let $\pi^+(x, y) = (\bar{x}, \bar{y})$. Then $E(x, y) = (\bar{x}, \bar{y})$, $W(x, y) = W(\bar{x}, \bar{y})$, $H(x, y) = H(\bar{x}, \bar{y})$. Since $f(x, y) = 0$, it follows that $| x |^2 = | \bar{x} |^2 = 2\alpha(H(x, y))$.

LEMMA 2.6. *π^+ admits a unique extension as a diffeomorphism of n^+ onto n^-.*

Proof. Recall that $E(x, y) = (y) \times W(x, y) - x \mid x \mid^{-1}$. Let $E(x, y) = E$ and $W(x, y) = W$. We can write

$$\bar{x} = (\bar{x} \cdot E) \mid E \mid^{-2} E + \bar{x} \cdot (E \times W) \mid E \times W \mid^{-2} E \times W.$$

We compute that $\bar{x} \cdot E = W \cdot W - | x |$, and $\bar{x} \cdot (E \times W) = -(\bar{x} \cdot \bar{y})(W \cdot W) = (x \cdot y)(W \cdot W)$. Since $\bar{x} \times \bar{y} = W = (x) \times (y)$, we can solve for $\bar{y}$, obtaining $\bar{y} = (W \times \bar{x}) \mid x \mid^{-2} + (x \cdot y) \mid x \mid^{-2} \bar{x}$. Thus the map π^+ sends $(x, y) \in n^+ - a^+$ to

$$\bar{x} = \mid E(x, y)\mid^{-2} [(W \cdot W - \mid x \mid) E(x, y) + (x \cdot y) E(x, y) \times W(x, y)]$$

and

$$\bar{y} = \mid x \mid^{-2} [W(x, y) \times \bar{x} + (x \cdot y) \bar{x}].$$

Notice that $\bar{x}(x, y)$ and $\bar{y}(x, y)$ are defined for all $(x, y) \in n^+$. Thus we extend π^+ to n^+ using the same formula. For $(x, y) \in a^+$ we have $W(x, y) = 0$, and it follows that $\bar{x} = x$, $\bar{y} = -y$. Thus π^+ is a diffeomorphism.

The lemma shows that the singularity S of the 2-body problem is regularizable. A natural step at this point is to give a topological characterization of the surfaces of constant energy contained in M'. These surfaces were classified by Kaplan [2]. He showed, in particular, that for the planar 2-body problem the 3-dimensional surfaces of constant negative energy were homeomorphic to real projective space P^3. In a forthcoming paper [3], J. Moser shows that the 2-body flow in n-dimensions on surfaces of constant negative energy is conjugate to the geodesic flow on the unit tangent bundle of S^n. Thus, rather than give a formal proof of these results from the present point of view, we will only give an informal argument.

We consider (x, y) now as belonging to $R^2 \times R^2$. Fix

$$H(x, y) = \tfrac{1}{2} \mid y \mid^2 - \mid x \mid^{-1} = h < 0.$$

The projection $p : (x, y) \to x$ takes the energy surface $\{H = h\}$ onto a punctured disk D in the x plane centered at $x = 0$ with radius $-(1/h)$. It is possible to choose a smaller disk $D_1 \subset D$ of radius $2\alpha(h)$ such that $p^{-1}(D_1)$ is an isolating block for the flow on the energy surface. $p^{-1}(D_1)$ is homeomorphic to a solid torus with the center line removed. It is easy to see that the set $\{H = h\} - p^{-1}(\text{int } D_1)$ is homeomorphic to a solid torus, the boundary of which corresponds to $p^{-1}(\partial D_1)$. Thus the regularized energy surface is a solid torus T with certain points on its boundary identified. In fact, n^+ and n^- are annuli on ∂T which twist once around T as shown in Fig. 1.

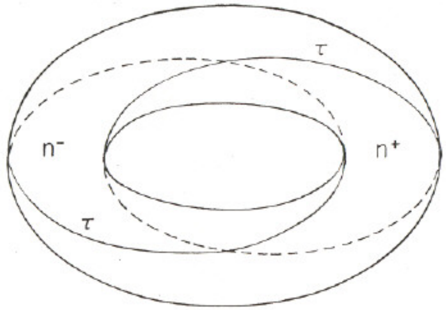

FIGURE 1

The mapping π^+ from n^+ to n^- is shown in Fig. 2.

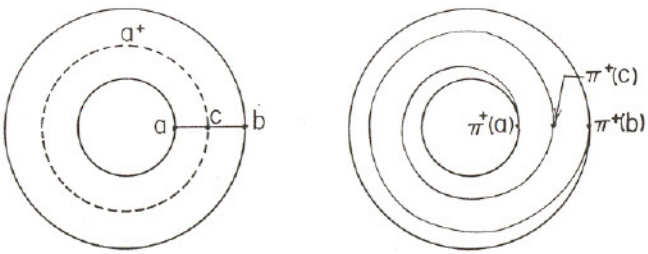

FIGURE 2

We can show that the regularized energy surface is homeomorphic to projective space P^3 as follows: Cut the solid torus along an annulus spanning τ obtaining two solid tori T_1 and T_2. The boundaries of T_1 and T_2 are identified as shown in Fig. 3. In particular, the curve $\gamma_1 \subset \partial T_1$ is identified with the curve $\gamma_2 \subset \partial T_2$.

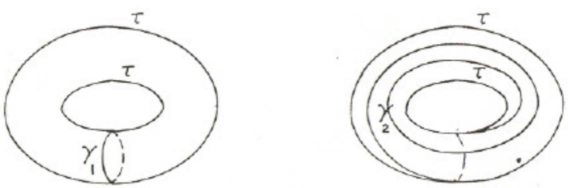

FIGURE 3

Now cut T_1 in the disk spanned by γ_1 and sew the boundary of T_1 to the boundary of T_2 thus obtaining a solid ball with antipodal points identified as shown in Fig. 4. (The Mobius band in T_2 bounded by γ_2 is sewn on the boundary of the solid ball.) This space we recognize as P^3.

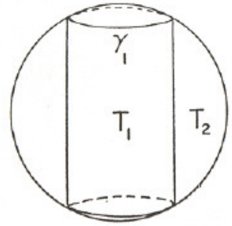

FIGURE 4

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