Dynkin's π-λ Theorem:

If $\mathscr{P} \subseteq \mathscr{L}$ for a π-system $\mathscr{P}$ and a λ-system $\mathscr{L}$, then

$$\sigma(\mathscr{P}) \subseteq \mathscr{L}.$$

Proof:

Define $\mathscr{L}_0$ to be the smallest λ-system containing $\mathscr{P}$. Then, by definition

$$\mathscr{P} \subseteq \mathscr{L}_0 \subseteq \mathscr{L}.$$

If we can show that $\mathscr{L}_0$ is a σ-field, we are done since $\sigma(\mathscr{P})$ is the smallest σ-field containing $\mathscr{P}$ so

$$\sigma(\mathscr{P}) \subseteq \mathscr{L}_0 \subseteq \mathscr{L}.$$

By the previous Lemma, since $\mathscr{L}_0$ is a λ-system, if we can show that $\mathscr{L}_0$ is also a π-system, we have shown that it is a σ-field.

Take any $A, B \in \mathscr{L}_0$. We must show that $A \cap B \in \mathscr{L}_0$.

- For any set $C \subseteq \Omega$, define the class of sets

  $$\mathscr{L}_C := \{D \subseteq \Omega : D \cap C \in \mathscr{L}_0\}.$$

  It is routine to verify that, $C \in \mathscr{L}_0 \Rightarrow \mathscr{L}_C$ is a λ-system. Please take a moment to convince yourelf of this.

- Suppose $A \in \mathscr{P}$. Then for every $C \in \mathscr{P}$, $A \cap C \in \mathscr{P}$ (since $\mathscr{P}$ is a π-system). Therefore, since $\mathscr{P} \subseteq \mathscr{L}_0$, $A \cap C \in \mathscr{L}_0$ which implies that $C \in \mathscr{L}_A$ and hence $\mathscr{P} \subseteq \mathscr{L}_A$ since this is true for every $C \in \mathscr{P}$.

- $A \in \mathscr{P} \subseteq \mathscr{L}_0 \Rightarrow \mathscr{L}_A$ is a λ-system. Since $\mathscr{L}_0$ is the smallest λ-system containing $\mathscr{P}$, we have that

  $$\mathscr{L}_0 \subseteq \mathscr{L}_A.$$

- By definition of $\mathscr{L}_A$, this gives us that $A \cap C \in \mathscr{L}_0$ for every $C \in \mathscr{L}_0$.

- Now consider our fixed $B \in \mathscr{L}_0$. We already know that $\mathscr{L}_B$ is a λ-system. As above, we can show that $\mathscr{P} \subseteq \mathscr{L}_B$ and therefore $\mathscr{L}_0 \subseteq \mathscr{L}_B$ since $\mathscr{L}_0$ is the smallest λ-system containing $\mathscr{P}$.

- So, for $A \in \mathscr{L}_0$, $A \in \mathscr{L}_B$ which implies that

  $$A \cap B \in \mathscr{L}_0,$$

  as desired. $\square$