Distributions and the Generalized Fourier Transform

With an eye for RBF applications
RBFs and the Fourier Transform

• Fourier transforms are really helpful for a lot of theoretical proofs of RBFs (eg, infinite lattice convergence, or positive definite-ness.

\[ \mathcal{F}f = \hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx \]

• In a multidimensional radially symmetric function \( \phi: \mathbb{R}^d \rightarrow \mathbb{R} \), \( \phi = \phi(\mathbf{r} = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}) \), we can find that the Fourier transform depends only on \( \rho = \sqrt{\omega_1^2 + \omega_2^2 + \cdots + \omega_d^2} \).

• This is called the Hankel transform,

\[ \hat{\phi}(\rho) = \frac{1}{(2\pi)^{d/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(||\mathbf{x}||)e^{-i\omega \cdot \mathbf{x}} \, d\mathbf{x} = \frac{1}{\rho^{d/2-1}} \int_0^\infty \phi(r)r^{d/2}J_{d/2-1}(r\rho) \, dr. \]

• Sometimes the Fourier series diverges.
The Schwartz Space

• The Schwartz Space (denoted $S$) is a set of functions on $\mathbb{R}^n$ which are smooth and rapidly decreasing.

• All functions $f \in C^\infty (\mathbb{R}^n)$ such that $\sup |x^\alpha \partial^\beta f (x)|$ is finite for all positive integers $\alpha, \beta$.

• Obviously the Fourier transform behaves nicely on $S$; it is a one-to-one and onto transformation whose inverse is its conjugate.
Tempered Distributions

- The topological dual of $S$ is the space of continuous linear functionals $T: S \to \mathbb{C}$. We call elements of this space tempered distributions.

- By the Riesz Representation Theorem, if $T(f)$ is a tempered distribution, we can find a function $T$ such that $T(f) = \langle T, f \rangle$.
  
  eg, $T(f) = f(0)$ is a continuous linear function on $S$, and may also be represented as $T = \delta$, the Dirac Delta.
Generalized Fourier Transform

- We define $\langle FT, f \rangle = \langle T, F^* f \rangle = \langle T, F f \rangle$ on $S^*$, so the Fourier transform of a tempered distribution $T$ is the distribution with identical action, but on the Fourier transform of the test function.

  e.g., $\langle F \delta, f \rangle = \langle \delta, F f \rangle = \hat{f}(0) = \frac{1}{(2\pi)^{d/2}} \langle 1, f \rangle$, so $\hat{\delta} = \frac{1}{(2\pi)^{d/2}}$, a constant.

- This definition allows us to inherit nice properties, like the adjoint-inverse relationship and convolution-product relationship.

- The proofs are a bit harder, but this generalizes by density and limit convergence to the superspace of general Distributions, which are the continuous linear functionals of smooth, compactly supported test functions.