Supplementary Notes on Discrete Fourier Analysis

The connections of discrete Fourier analysis to interpolation and quadrature are found in the notes on interpolation and quadrature, respectively. These notes deal directly with discrete Fourier vectors, the discrete Fourier transform (DFT), and diagonalization of circulant matrices, including finite difference matrices in a periodic domain.

1 The k^{th} discrete Fourier vector is denoted $\boldsymbol{v}^k$; it has elements

$$\boldsymbol{v}^k_{j+1} = e^{2\pi \mathrm{i} jk/N}, \; j = 0, \ldots, N-1$$

where $\mathrm{i} = \sqrt{-1}$. Note the indexing convention!

2 There are only N distinct Fourier vectors of length N because $\boldsymbol{v}^k = \boldsymbol{v}^{k+\ell N}$ for any integer ℓ:

$$\boldsymbol{v}^{k+\ell N}_{j+1} = e^{2\pi \mathrm{i} j(k+\ell N)/N} = e^{2\pi \mathrm{i} jk/N} e^{2\pi \mathrm{i}\ell} = e^{2\pi \mathrm{i} jk/N} = \boldsymbol{v}^k_{j+1}.$$

The standard convention is to use k from $-N/2+1$ to $N/2$ if N is even, and from $-(N-1)/2$ to $(N-1)/2$ is N is odd.

3 The discrete Fourier vectors have the convenient property that $\boldsymbol{v}^k_{j+1+\ell N} = \boldsymbol{v}^N_{j+1}$ for any integer ℓ:

$$\boldsymbol{v}^k_{j+1+\ell N} = e^{2\pi \mathrm{i}(j+\ell N)k/N} = e^{2\pi \mathrm{i} jk/N} e^{2\pi \mathrm{i}\ell k} = e^{2\pi \mathrm{i} jk/N} = \boldsymbol{v}^k_{j+1}.$$

We'll make a periodic extension of any vector by defining $\boldsymbol{x}_{j+N} = \boldsymbol{x}_j$ for any j.

4 The discrete Fourier vectors are orthogonal with respect to the standard complex dot product:

$$\boldsymbol{v}^k \cdot \boldsymbol{v}^\ell = \sum_{j=0}^{N-1} e^{-2\pi \mathrm{i} jk/N} e^{2\pi \mathrm{i} j\ell/N} = \sum_{j=0}^{N-1} \left(e^{2\pi \mathrm{i}(\ell-k)/N}\right)^j$$

This is a geometric series (of finite length), so we can use the formula

$$\boldsymbol{v}^k \cdot \boldsymbol{v}^\ell = \sum_{j=0}^{N-1} \left(e^{2\pi \mathrm{i}(\ell-k)/N}\right)^j = \frac{1 - \left(e^{2\pi \mathrm{i}(\ell-k)/N}\right)^N}{1 - e^{2\pi \mathrm{i}(\ell-k)/N}}.$$

When $k \neq \ell$ the numerator is 0. When $k = \ell$ the formula does not apply; rather, the sum is simply a sum of N ones. So the discrete Fourier vectors are orthogonal, and have norm $\sqrt{N}$.

5 Since the discrete Fourier vectors are orthogonal they form a basis for $\mathbb{C}^N$. Computing the coefficients of a vector in this basis is the same as computing the DFT of the vector. Computing the DFT of a vector is the same as multiplying that vector by the following matrix

$$\mathbf{F} = \frac{1}{N}[\boldsymbol{v}^0, \boldsymbol{v}^1, \ldots, \boldsymbol{v}^{N-1}]^*$$

where the superscript * denotes the complex conjugate transpose. The FFT is simply a fast algorithm for applying this matrix to vectors. Note that the standard convention is to order the columns from $\boldsymbol{v}^0$ to $\boldsymbol{v}^{N-1}$, even though we've seen that $\boldsymbol{v}^{N-1} = \boldsymbol{v}^{-1}$ (and similarly for the other columns).

6 Consider the 'shift' matrix

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Note the following two properties: (i) $(\mathbf{N}\boldsymbol{x})_j = \boldsymbol{x}_{j+1}$ (using our periodic extension convention so that $\boldsymbol{x}_{N+1} = \boldsymbol{x}_1$), and (ii) $\mathbf{N}^2$ has the same structure as $\mathbf{N}$, but all the entries are shifted (circularly) to the right (or up). I.e.

$$\mathbf{N}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & \ddots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

$\mathbf{N}^p$ is similar.

Notice that the discrete Fourier vectors are eigenvectors of $\mathbf{N}$:

$$\left(\mathbf{N}\boldsymbol{v}^k\right)_{j+1} = \boldsymbol{v}^k_{j+2} = e^{2\pi \mathrm{i}(j+1)k/N} = (e^{2\pi \mathrm{i}k/N})e^{2\pi \mathrm{i}jk/N} = (e^{2\pi \mathrm{i}k/N})\boldsymbol{v}_{j+1}$$

$$\text{i.e. } \mathbf{N}\boldsymbol{v}^k = (e^{2\pi \mathrm{i}k/N})\boldsymbol{v}^k$$

so the eigenvalue is $e^{2\pi \mathrm{i}k/N}$.

7 A circulant matrix has the following form

$$\mathbf{C} = a_0\mathbf{I} + a_1\mathbf{N} + a_2\mathbf{N}^2 + \ldots + a_{N-1}\mathbf{N}^{N-1}.$$

Since circulant matrices are polynomials in $\mathbf{N}$, the eigenvectors of circulant matrices are also the discrete Fourier vectors, and the eigenvalues are

$$a_0 + a_1 e^{2\pi \mathrm{i}k/N} + a_2(e^{2\pi \mathrm{i}k/N})^2 + \ldots + a_{N-1}(e^{2\pi \mathrm{i}k/N})^{N-1}.$$

Notice that this is just the dot product of the first row of $\mathbf{C}$ with $\boldsymbol{v}^k$.

8 Finite difference matrices on an equispaced periodic domain are circulant matrices. For example, the first-order forward difference approximating ∂_x is

$$\frac{1}{h}\left[\mathbf{N} - \mathbf{I}\right]$$

while the second-order centered difference approximating ∂_x^2 is

$$\frac{1}{h^2}\left[\mathbf{N} - 2\mathbf{I} + \mathbf{N}^{-1}\right].$$

We can find the eigenvalues of these matrices using the above methods. For example, the eigenvalues of the second-order centered difference are

$$\frac{e^{2\pi \mathrm{i}k/N} - 2 + e^{-2\pi \mathrm{i}k/N}}{h^2} = -\frac{4}{h^2}\sin^2\left(\frac{k\pi}{N}\right).$$

This is the heart of von Neumann analysis. In von Neumann analysis you write the spatially-discrete PDE as

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \mathbf{L}\boldsymbol{u} \text{ or } \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} = \mathbf{L}\boldsymbol{u} \text{ (or similar, depending on which PDE you've discretized)}$$

Then, if the grid is equispaced and you're using finite differences then the matrix $\mathbf{L}$ can be diagonalized using the discrete Fourier vectors, and the eigenvalues are easy to compute. Sometimes the term 'von Neumann analysis' refers to analysis of a fully-discrete (space & time) PDE, i.e. you apply some method like Forward Euler to get a system of linear difference equations (instead of the above linear system of ODEs), and then seek solutions where the spatial component is a discrete Fourier vector and the time component is exponential, e.g.

$$\boldsymbol{u}(n\delta t) = r_k^n \boldsymbol{v}^k.$$