

DIMENSIONAL ANALYSIS AND SCALING

Introduction

Most problems are first formulated in physical units (such as meters, kilograms, seconds etc. in case of the 'metric system', else maybe in 'imperial units' or even in mixed units, say nautical miles for distances). In some cases, units can be used throughout the solution process. However, there are often strong reasons to 'non-dimensionalize' a problem - change dependent and independent variables in such a way that the equations become entirely independent of the choice of measuring units. The primary reasons for this are:

1. The form of the equations becomes often much simpler,
2. The process will often reveal that many apparently independent free parameters can be combined into much fewer ones. Examples from fluid mechanics include Mach number, Reynolds number, Prandtl number, etc. To fully explore the full parameter space for all possible solutions, it may prove sufficient to vary only one or two parameters instead of dealing with many more parameters,
3. A further procedure called *scaling* allows equations to be brought to forms which are well suited for subsequent *perturbation analysis* - a very powerful collection of analytical techniques to find approximate solutions to problems impossible to solve analytically and difficult and/or expensive to solve numerically.

One of the great strengths of dimensional analysis is that it can be performed without knowing the governing equations for a problem. The process of *scaling* requires also physical insights, and it can greatly simplify and clarify governing equations. It allows us to clearly separate phenomena according to their scales, and is often a necessary first step in applying perturbation methods.

We will describe the two procedures mainly through a series of example, starting with dimensional analysis:

Dimensional analysis (without governing equations available)

Example 1: A swinging pendulum:

A mass is attached at the end of a rod that has length L and is of negligible weight. We assume that the period P depends only on L and on the acceleration of gravity g . The task is to simplify the expression $P = P(L, g)$.

Solution:

With \mathcal{L} denoting the 'dimension' of length and \mathcal{T} the 'dimension' of time, the type of units which would describe the variable P and parameters L and g of our problem become

Variable	Dimension
P pendulum period	\mathcal{T}

Parameters	Dimension
L pendulum length	\mathcal{L}
g acceleration of gravity	$\mathcal{L}\mathcal{T}^{-2}$

The only way we can combine powers of $L \sim \mathcal{L}$ and $g \sim \mathcal{L}\mathcal{T}^{-2}$ in order to obtain the dimension of $P \sim \mathcal{T}$ is $L^{1/2} g^{-1/2} \sim \mathcal{T}$, telling that P must be a function of $\sqrt{\frac{L}{g}}$. We can write this as $P = P\left(\sqrt{\frac{L}{g}}\right)$. Instead of having to find a function of two variables $P = P(L, g)$, we have simplified the relationship to a function of one variable.

The derivation did not require us to have any governing equation available for the motion, but if we do, we can use it to confirm the result. With $\theta(t)$ denoting the angle the pendulum deviates from vertical, its governing equation becomes

$$\theta''(t) = -\frac{g}{L} \sin \theta(t)$$

We assumed the period P to depend only on L and g . This becomes the case when θ is small, i.e. when $\sin \theta \approx \theta$. The solution to the simplified equation $\theta''(t) = -\frac{g}{L}\theta(t)$ is

$$\theta(t) = c_1 \cos \sqrt{\frac{g}{L}} t + c_2 \sin \sqrt{\frac{g}{L}} t ,$$

i.e. the period is $P = 2\pi\sqrt{\frac{L}{g}}$, in complete agreement with the form we derived above.

With more parameters in a problem, one needs a more systematic approach.

Example 2: Fluid drag on a sphere.

Assume the drag force d on a sphere which is traveling through a fluid depends on D (its diameter), U (its velocity), ρ (the fluid density) and μ (the fluid viscosity). The task is to simplify the relation

$$d = f(D, U; \rho, \mu). \tag{1}$$

Solution:

The quantities and their corresponding dimensions are

Variable	Dimension
d drag force	$\mathcal{M}\mathcal{L}/\mathcal{T}^2$

Parameters	Dimension
D sphere diameter	\mathcal{L}
U fluid velocity	\mathcal{L}/\mathcal{T}
ρ fluid density	$\mathcal{M}\mathcal{L}^3$
μ fluid viscosity	$\mathcal{M}/\mathcal{L}\mathcal{T}$

We can readily see that for ex. D , U , and ρ are dimensionally independent. Among these three variables, \mathcal{T} appears only in U and \mathcal{M} only in ρ - there is therefore no possibility to get complete cancellation of dimensions by forming a product $D^\alpha U^\beta \rho^\gamma$ (unless $\alpha = \beta = \gamma = 0$). However, we can note that

$$d \sim \mathcal{M}\mathcal{L}/\mathcal{T}^2 \sim \rho U^2 D^2, \quad (2)$$

$$\mu \sim \mathcal{M}/\mathcal{L}\mathcal{T} \sim \rho U D. \quad (3)$$

Based on (2) and (3), the quantities $\Pi_0 = \frac{d}{\rho U^2 D^2}$ and $\Pi_1 = \frac{\mu}{\rho U D}$ are dimensionless. Equation (1) can therefore be written

$$\Pi_0 = \frac{d}{\rho U^2 D^2} = f(D, U, \rho, \Pi_1) \quad (4)$$

Here the "f" denotes a generic function - not the same in (4) as in (1). The key point now is the *Π -theorem* (or *Pi-theorem*): Since the LHS of (4) is purely non-dimensional, so must the RHS be, i.e. the function f cannot depend explicitly on either D , U , or ρ . We obtain

$$\Pi_0 = f(\Pi_1).$$

The unknown function in (1) has been simplified from having four arguments to a function having just one argument. The quantities that we have denoted as Π_0 and Π_1 arise sufficiently often in fluid dynamics to have their own names. Π_0 is known as the drag coefficient $C_d = \frac{d}{\rho U^2 D^2}$ and $1/\Pi_1$ is known as the Reynolds number $Re = \frac{\rho U D}{\mu}$. The relation

$$C_d = f(Re)$$

requires only one single function to be determined in order to find the drag for any values of D , U , ρ and μ .

The Π -theorem can be stated in much more general (and abstract) form as follows:

Π -theorem:

A physical relationship between a dimensional quantity and its governing dimensional parameters can be written as a relationship between problem's dimensionless parameters. The total number of dimensionless parameters is equal to the total number of governing parameters minus the number of governing parameters with independent dimensions.

In mathematical terms, given a variable w with a_1, a_2, \dots, a_n as governing parameters of independent dimension and b_1, b_2, \dots, b_m as governing parameters of dependent dimension, then

$$w = f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m)$$

implies

$$\begin{aligned} \Pi_0 &= \frac{w}{a_1^{p_1} a_2^{p_2} \dots a_n^{p_n}} = f\left(\frac{b_1}{a_1^{q_1} a_2^{q_2} \dots a_n^{q_n}}, \frac{b_2}{a_1^{r_1} a_2^{r_2} \dots a_n^{r_n}}, \dots, \frac{b_m}{a_1^{s_1} a_2^{s_2} \dots a_n^{s_n}}\right) \\ &= f(\Pi_1, \Pi_2, \dots, \Pi_m) \end{aligned}$$

for certain sets of constants $p_i, q_i, \dots, s_i, i = 1, \dots, n$.

Example 3:

A projectile of mass m is shot straight up from the earth's surface with initial velocity v_0 . Maybe unrealistically, we assume that air resistance and effects due to the earth's rotation can be ignored, but *not* the decrease of gravity at increased heights. The task is to find an expression for y_{max} , the maximum height that is reached.

Solution:

The formula for the maximum height y_{max} will be of the form

$$y_{max} = f(g, v_0, R)$$

where

Variable	Dimension
y_{max} maximum height	\mathcal{L}
Parameters	Dimension
R radius of earth	\mathcal{L}
v_0 initial velocity	\mathcal{L}/\mathcal{T}
g acceleration at surface	$\mathcal{L}/\mathcal{T}^2$

Of the three parameters R , v_0 , and g , only two are linearly independent. We can express each one in terms of the others:

$$R \sim \mathcal{L} = (\mathcal{L}^2/\mathcal{T}^2) / (\mathcal{L}/\mathcal{T}^2) \sim \frac{v_0^2}{g},$$

$$v_0 \sim \mathcal{L}/\mathcal{T} = (\mathcal{L}^{1/2}/\mathcal{T}) \mathcal{L}^{1/2} \sim g^{1/2} R^{1/2},$$

$$g \sim \mathcal{L}/\mathcal{T}^2 = (\mathcal{L}/\mathcal{T})^2 / \mathcal{L} \sim \frac{v_0^2}{R}.$$

In order to express y_{max} in terms of the parameters, we note that it has dimension \mathcal{L} . So we have two options:

1. $R \sim \mathcal{L}$

2. $\frac{v_0^2}{g} \sim \mathcal{L}$

In the Case 1, we need to choose another independent variable. Hence Case 1 separates in two sub-cases:

1 a. Take R and v_0 as independent parameters. We can then write

$$\Pi_0 = \frac{y_{max}}{R} = f(R, v_0, \Pi_1)$$

with $\Pi_1 = \frac{g}{v_0^2 R^{-1}} \left(= \frac{gR}{v_0^2} \right)$ and, by the Π -theorem, the explicit dependence of f on R and v_0 is absent, i.e.

$$\Pi_0 = \frac{y_{max}}{R} = f(\Pi_1) \quad (5)$$

1 b. Take R and g as independent parameters. Similarly, we get

$$\Pi_0 = \frac{y_{\max}}{R} = f(R, g, \Pi_1)$$

with $\Pi_1 = \frac{v_0^2}{gR}$, i.e. we conclude that

$$\Pi_0 = \frac{y_{\max}}{R} = f(\Pi_1). \quad (6)$$

The only difference from the previous case that the definition of Π_1 is inverted.

2. Take v_0^2 and g as independent parameters. We now get

$$\Pi_0 = \frac{y_{\max}}{v_0^2/g} = f(v_0, g, \Pi_1)$$

with $\Pi_1 = \frac{R}{v_0^2/g}$ and the explicit dependence of f on v_0 and g vanishing:

$$\Pi_0 = \frac{y_{\max}}{v_0^2/g} = f(\Pi_1) \quad (7)$$

All the cases 1 a, 1 b, and 2 have in common that Π_1 is the same (or inverted).

Like for the pendulum, we can verify some of these simplifications if we have the governing equation for the flight of the projectile.

From Newton's law Force = mass \times acceleration and his law of gravity (force decaying inversely with the square of the distance - here to the center of the earth) follow

$$\frac{d^2y}{dt^2} = - \frac{gR^2}{(y+R)^2} \quad \text{with initial conditions } y(0) = 0, \frac{dy}{dt}(0) = v_0. \quad (8)$$

Although equation (8) - with considerable difficulty - can be solved in closed form, almost any minor alteration to it (such as addition of another term) will make it unsolvable in terms of elementary functions. However, some quantities such as y_{\max} can nevertheless be obtained quite easily. Multiplying (8) by $\frac{dy}{dt}$ and integrating gives $\frac{1}{2}(\frac{dy}{dt})^2 = \frac{2gR^2}{y+R} + C$. From the initial condition $\frac{dy}{dt}(0) = v_0$ and the fact that $\frac{dy}{dt} = 0$ at the maximal height follows, after brief algebra, that $y_{\max} = R \cdot \frac{1}{2(\frac{gR}{v_0^2}) - 1}$. This form agrees

immediately with (5), but can equally well be re-cast into (6) or (7).

The dimensional analysis has led to significant insights in how different quantities depend on each other, without any knowledge about actual governing equations. In cases when governing

equations are available, the additional procedure of *scaling* allows us to alter the equations in ways that simplify both analysis and computing.

Dimensional analysis (with governing equations available)

We consider again the projectile problem, as governed by equation (8).

Step 1: List the variables and parameters, and note what type of units they are expressed in:

Variables		Dimension
y	dependent variable	\mathcal{L}
t	independent variable	\mathcal{T}

Parameters		Dimension
R	radius of earth	\mathcal{L}
v_0	initial velocity	\mathcal{L}/\mathcal{T}
g	acceleration at surface	$\mathcal{L}/\mathcal{T}^2$

Step 2: For each of the two variables, find some combination of the parameters that produce the same dimensions. For example

y has dimension \mathcal{L} ; this is the same as that of R , and
 t has dimension \mathcal{T} ; the combination R/v_0 becomes also \mathcal{T} .

These choices were not unique. Although they will allow us to non-dimensionalize the equation, we will soon see that we did not get the equation in its most practical form.

Based on these choices, we introduce new (dependent and independent) variables

$$y^* = \frac{y}{R} \quad \text{and} \quad t^* = \frac{t}{R/v_0} \quad (9)$$

These variables are now non-dimensional, i.e. their numerical value do not depend on what kind of units were originally used (e.g. meters, inches, nautical miles or light years for distances and seconds, weeks or millennia for time).

Step 3: Express the original equation (8) in the new variables (from Step 2). To do this, we note first how a derivative changes if its dependent and independent variables are *scaled* by the changes of variables in (9):

If $y \rightarrow \alpha y^*$, $t \rightarrow \beta t^*$, then $\frac{dy}{dt} \rightarrow \frac{\alpha}{\beta} \cdot \frac{dy^*}{dt^*}$, $\frac{d^2y}{dt^2} \rightarrow \frac{\alpha}{\beta^2} \cdot \frac{d^2y^*}{d(t^*)^2}$, etc.; in general $\frac{d^p y}{dt^p} \rightarrow \frac{\alpha}{\beta^p} \cdot \frac{d^p y^*}{d(t^*)^p}$. The placement of the p 's in the (curious) notation $\frac{d^p y}{dt^p}$ for a p 'th derivative makes this easy to remember - the scale change for the variable in the numerator just factors right out, whereas the one for the variable in the denominator comes out raised to the p 'th power.

After slight algebra, the change of variables (7) and simplifying notation by dropping the asterisks on y^* and t^* , gives

$$\varepsilon \frac{d^2 y}{dt^2} = -\frac{1}{(y+1)^2}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1 \quad (10)$$

where

$$\varepsilon = \frac{v_0^2}{gR} \quad (11)$$

As we noted, step 2 was not unique. Instead of

" t^* has dimension \mathcal{T} ; the combination R/V becomes also \mathcal{T} ",

we could have noted

" t^* has dimension \mathcal{T} ; the combination $\sqrt{R/g}$ becomes also \mathcal{T} .

Instead of arriving at (10), this alternative non-dimensionalization gives

$$\frac{d^2 y}{dt^2} = -\frac{1}{(y+1)^2}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = \varepsilon^{1/2}, \quad (12)$$

with ε unchanged from (11). Neither version (10) nor (12) will turn out to be suitable for perturbation analysis. By adding physical insights, the process of scaling resolves the ambiguity, and leads to equations that are well suited for perturbation (small ε) analysis.

Scaling

Scaling addresses the ambiguity we found in how to non-dimensionalize the dependent and independent variables (here y and t). Although both of the versions we considered just above were successful in simplifying the ODE and in identifying how the independent parameters could be consolidated into one single one ($\varepsilon = \frac{v_0^2}{gR}$), they are unsatisfactory for example if we want to apply the very powerful techniques known as perturbation methods. These methods allow the

present, difficult-to-solve ODE, to be replaced by a series of simpler ones, whose solutions provide successive terms $y_k(t)$ in expansions of the form

$$y(t, \epsilon) = \sum_{k=0}^N \epsilon^k y_k(t)$$

A useful expansion of this type requires, as a minimum, that the solution for $\epsilon = 0$ makes sense as a limit of $\epsilon \rightarrow 0$. Putting $\epsilon = 0$ gives in the first case (8) the impossible requirement

$$0 = -\frac{1}{(y+1)^2}, \quad y(0) = 0,$$

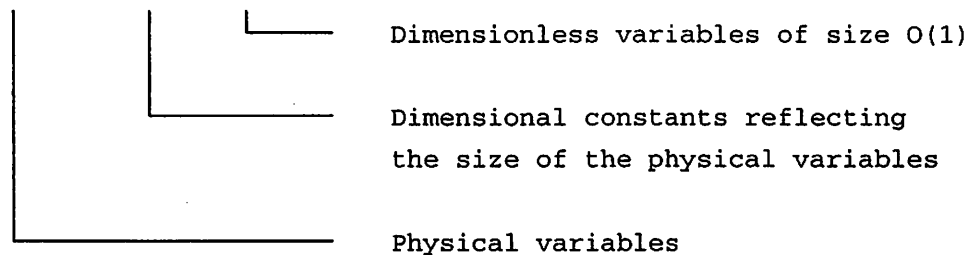
and in the second case (10)

$$\frac{d^2y}{dt^2} = -\frac{1}{(y+1)^2}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 0$$

implying that $y(t) < 0$ for all $t > 0$, i.e. the projectile will never rise at all. Both cases are obviously nonsensical. The problem is that, as $\epsilon \rightarrow 0$, the non-dimensionalization also causes y and t to change in such ways that the time- and space ranges we are interested in get stretched out or compressed to nothing, rather than converging to some fixed ranges.

We need the non-dimensionalization to have the following structure:

$$\begin{aligned} y &= c_y \cdot y^* \\ t &= c_t \cdot t^* \end{aligned}$$



This requires some physical arguments:

1. *Decide what parameter regime we are interested in:*

At present, this is v_0 small, i.e. ϵ small,

2. *In the simplified case of ε small, find dimensional constants that are proportional to the physical variables:*

With v_0 small, the projectile will not reach very high, and the acceleration of gravity stays approximately constant. We can then easily solve for the height that will be reached ($\frac{1}{2}v_0^2/g$) and the time this will take (v_0/g for each of the up- and down paths). Not needing to be concerned about numerical constants, this suggests choosing $c_y = v_0^2/g$ and $c_t = v_0/g$.

3. *Carry out the change of variables:*

$$\text{We get } \frac{d^2y}{dt^2} = -\frac{1}{(1+\varepsilon y)^2}, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1 \quad (13)$$

(as before with $\varepsilon = \frac{v_0^2}{gR}$).

We have now a formulation in which letting $\varepsilon \rightarrow 0$ causes $y(t,\varepsilon)$ to converge to a physically relevant limit. The equation (13) is ideally suited for perturbation analysis.

There are several different but related approaches to scaling, and it depends a lot on the background of the practitioner which one is preferred. Hence:

Alternative argument for scaling the projectile problem:

Writing (8) in the form $y'' = \frac{-g}{\left(\frac{y}{R} + 1\right)^2}$, we first note that $y \ll R$ so, to a first

approximation, $y'' = -g$. The key question in perturbation analysis is to quantify how much less y is than R , so we can make successive corrections to this first approximation.

The only force exerted on the projectile is that of gravity. Hence

$$y'' = O(g) \quad (\text{meaning that the scale of } y'' \text{ is } g) \quad (14)$$

From the initial condition

$$y_t(0) = v_0 \quad (15)$$

we know that

$$\frac{d}{dt} y_t(0) = O\left(\frac{v_0}{T}\right)$$

(because $\frac{d}{dt} = O\left(\frac{1}{T}\right)$) which has to match the acceleration

$$\underbrace{\frac{d}{dt} y_t}_{IC} = \underbrace{y_{tt}}_{accel.}$$

Hence $O\left(\frac{v_0}{T}\right) = O(g)$, and

$$T = O\left(\frac{v_0}{g}\right) \tag{16}$$

From (14) and (15)

$$y_{tt} = O\left(\frac{g^2}{v_0^2} c_y\right) = O(g)$$

where c_y is the scale of the height. Solving for c_y , we get $O(c_y) = O\left(\frac{v_0^2}{g}\right)$. Thus, we have all our scales; t and y by only using the argument that gravity is the only force that accelerates the projectile. To conclude this analysis:

$$\begin{aligned} y &= c_y y^* & , & & t &= T t^* \\ &= \frac{v_0^2}{g} y^* & & & &= \frac{v_0}{g} t^* \end{aligned}$$

and

$$\frac{g^2}{v_0^2} \left(\frac{v_0^2}{g}\right) \frac{d^2 y^*}{dt^{*2}} = \frac{-g}{\left(\left(\frac{v_0^2}{gR}\right) y^* + 1\right)^2} ,$$

in agreement with (13).

It is quite common that equations are encountered in an already non-dimensionalized form. A first task is then to check whether some re-scaling is needed, or if $\varepsilon = 0$ indeed corresponds to $\varepsilon \rightarrow 0$. It is also common that no single re-scaling is appropriate for all parts of the spatial domain. One such case of 'singular' perturbation concerns boundary layers in fluids. Different scalings need to be used at a boundary and throughout the bulk of the domain. Separately scaled perturbation expansions from different regions have to be 'matched' to agree where they spatially meet.