Strong and Weak Equilibria for Time-Inconsistent Stochastic Control

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Outline

Introduction

- Why the *stronger* concept?

The Model

- Continuous-time Markov chain.

Main Results

- Characterizations of weak and strong equilibria.
- Existence of weak and strong equilibria.

Examples

Classical Stochastic Control

- Consider a controlled Markovian process X^α.

Stochastic Control

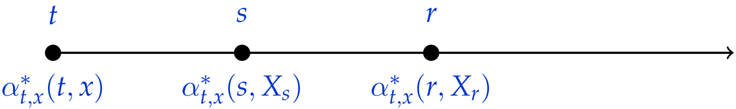
Given $(t,x) \in [0,\infty) \times \mathbb{R}^d$, can we solve

$$\sup_{\alpha \in \mathcal{A}} F(t,x,\alpha)? \tag{1}$$

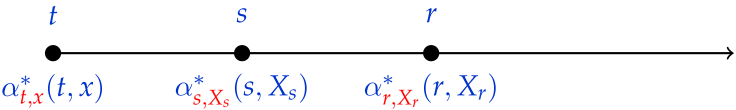
- **Classical Control Theory:**
  - **Want:** find an optimal control $\alpha^*_{t,x} \in \mathcal{A}$.
  - **Methods:** *dynamic programming*, *martingale approach*,...
  - Consider $\alpha^*_{t,x}$ as a mapping:

$$(t,x) \quad \longmapsto \quad \alpha^*_{t,x} \in \mathcal{A}.$$

- **Problem Solved.** *Feeling Good?*



- **The Reality:**



- **Time Inconsistency:**
  - $\alpha^*_{t,x}$, α^*_{s,X_s}, α^*_{r,X_r} may all be different.
  - The original objective (1) cannot be attained...

Time-inconsistent objectives:

- Non-exponential discounting:

$$F(t,x,\alpha) := \mathbb{E}_{t,x}[\delta(T-t)g(X_T^\alpha)].$$

- Payoff depending on initials (t,x):

$$F(t,x,\alpha) := \mathbb{E}_{t,x}[g(t,x,X_T^\alpha)].$$

- Nonlinear functionals of $\mathbb{E}[\cdot]$:

$$F(t,x,\alpha) := \mathbb{E}_{t,x}[g(X_T^\alpha)] - H(\mathbb{E}_{t,x}[g(X_T^\alpha)]).$$

- Probability distortion:

$$F(t,x,\alpha) := \int_0^\infty w\Big(\mathbb{P}_{t,x}\left[g(X_T^\alpha) > u\right]\Big)du.$$

How to resolve time inconsistency?

Consistent Planning [Strotz (1955-56)]

- Take into account future selves' behavior.

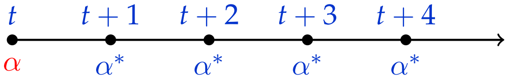
  Find an *equilibrium* strategy that

  once being enforced over time,
  no future self would want to deviate from.

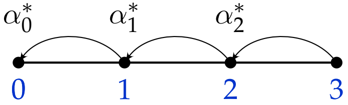
DISCRETE TIME

- **Definition:** $\alpha^* \in \mathcal{A}$ is an equilibrium if

$$F(t, x, \alpha^*) \geq F(t, x, \alpha \otimes_{t+1} \alpha^*), \quad \forall (t, x), \alpha.$$



- **How to find an equilibrium?**
  Backward sequential optimization [Pollak (1968)]:

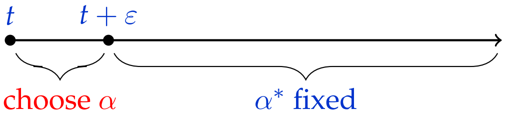


  - Limitation: Infinite horizon?

Continuous Time

- **Definition** (Ekeland & Lazrak (2006)):
  α^* is an *equilibrium* if

$$\liminf_{\varepsilon \to 0} \frac{F(t,x,\alpha^*) - F(t,x,\alpha \otimes_{t+\varepsilon} \alpha^*)}{\varepsilon} \geq 0 \quad \forall (t,x),\ \alpha.$$



- **How to find an equilibrium?**
  Ekeland & Pirvu (2008) characterize equilibrium α^* by a system of nonlinear differential equations (extended HJB system).

SUBSEQUENT STUDIES

- **Control problems:**

  A long list...

  Ekeland, Mbodji, & Pirvu (2012), Björk, Murgoci, & Zhou (2014), Dong & Sircar (2014), Björk & Murgoci (2014), Yong (2012), Björk, Khapko & Murgoci (2017), ...

- **Stopping problems:**

  Only two preprints...

  Ebert, Wei & Zhou (2017), Christensen & Lindensjö (2017).

  - transform stopping problem into control problem;
  - use the same definition and extended HJB system as in the control case.

The Problem

$$\liminf_{\varepsilon\to 0} \frac{F(t,x,\alpha^*) - F(t,x,\alpha \otimes_{t+\varepsilon} \alpha^*)}{\varepsilon} \geq 0 \quad \forall (t,x),\ \alpha. \qquad (2)$$

- This definition NOT fully making sense economically!
  - **Intuitively we want:**

    As $\varepsilon > 0$ small, it's better to stay with α^*.
  - **However**, there may exist α^* satisfying
    - for some (t,x), α,

      $$F(t,x,\alpha^*) < F(t,x,\alpha \otimes_{t+\varepsilon} \alpha^*) \quad \forall \varepsilon > 0 \text{ small;}$$
    - the limit in (2) is 0.
  - $\Longrightarrow$ (2) may include controls we don't really want...
    (2) may be too weaker a definition for an equilibrium.
- cf. Remark 3.5 of Björk, Khapko & Murgoci (2017).

IN THIS TALK...

- **New definition of continuous-time equilibria:**

  α^* is a strong equilibrium if for any (t, x) and α, there is $\varepsilon^*(t, x, \alpha) > 0$ such that

$$F(t, x, \alpha^*) \geq F(t, x, \alpha \otimes_{t+\varepsilon} \alpha^*), \quad \forall 0 < \varepsilon < \varepsilon^*. \tag{3}$$

  - Precise economic interpretation:

    If (3) is violated, agent at (t, x) has incentive to deviate to α in a however small interval $[0, \varepsilon]$.
  - A similar notion in Appendix D of He & Jiang (2017).
- **Relation between strong and weak equilibria**
- **Weak equilibria that are not strong**

The Model

- X: time-homogeneous continuous-time Markov chain.
- State space $S := \{1, 2, ..., N\}$.
- The generator $Q \in \mathbb{R}^{N \times N}$ of X is to be controlled.
  - Q_i: the i^{th}-row of Q.
  - D_i: admissible set for Q_i.

$$Q_i \in D_i \subseteq E_i := \left\{ (q_1, ..., q_N) \in \mathbb{R}^N : q_j \geq 0,\ j \neq i,\ q_i = -\sum_{j \neq i} q_i \right\}.$$

  - The control space:

$$\mathcal{Q} := \left\{ Q \in \mathbb{R}^{N \times N} : Q_i \in D_i,\ \forall i \in S \right\}.$$

The Model

- The objective:

$$F(i,Q) := \mathbb{E}_i\left[\int_0^\infty f(t, X_t, Q_{X_t})dt\right].$$

  - $\mathbb{E}_i$: expectation conditioned on $X_0 = i$.
  - always restart from time 0

    $\Longrightarrow$ t in $f(t,\cdot,\cdot)$ is *not* calendar time, *but* time difference.

    $\Longrightarrow$ the usual **time-homogeneous** setting.

- Typical example:

$$F(i,Q) := \mathbb{E}_i\left[\int_0^\infty \delta(t)g(X_t, Q_{X_t})dt\right],$$

where $\delta : [0,\infty) \to [0,1]$ is a discount function.

Integrability Condition

- Assume

$$\int_0^\infty \sup_{i\in S} |f(t,i,Q_i)|dt < \infty, \quad \forall Q \in \mathcal{Q}. \tag{4}$$

- Non-exponential discounting: (4) reduces to

$$\int_0^\infty \delta(t)dt < \infty. \tag{5}$$

  - *Hyperbolic:* $\delta(t) := \frac{1}{1+\beta t}$, $\beta > 0$, violates (5).
  - *Generalized hyperbolic:* $\delta(t) := \frac{1}{(1+\beta t)^k}$, $\beta > 0$ and $k > 1$, satisfies (5).
  - *Pseudo-exponential:* $\delta(t) := \lambda e^{-\rho t} + (1-\lambda)e^{-\rho' t}$, $\lambda \in (0,1)$ and $\rho, \rho' > 0$, satisfies (5).

Weak Equilibria

$Q^* \in \mathcal{Q}$ is a weak equilibrium, if

$$\liminf_{\varepsilon \to 0} \frac{F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*)}{\varepsilon} \geq 0 \quad \forall i \in S,\ Q \in \mathcal{Q}. \tag{6}$$

Strong Equilibria

$Q^* \in \mathcal{Q}$ is a strong equilibrium, if for any $i \in S$ and $Q \in \mathcal{Q}$, there exists $\varepsilon(i, Q) > 0$ such that

$$F(i, Q^*) \geq F(i, Q \otimes_{\varepsilon'} Q^*) \quad \forall 0 < \varepsilon' \leq \varepsilon. \tag{7}$$

By definition,

- A strong equilibrium is weak;
- If (6) holds with strict equality for all $i \in S$ and $Q \in \mathcal{Q}$, the weak equilibrium Q^* is also strong.

Conditions

Assume

1) $t \mapsto f(t, i, \mathbf{q})$ is $\mathcal{C}_1$ on $[0, \infty)$, for all $i \in S$ and $\mathbf{q} \in D_i$.

   - Consider 1st-order residual function $r(t, \varepsilon; i, \mathbf{q})$, i.e.

$$|f(t+\varepsilon, i, \mathbf{q}) - (f(t, i, \mathbf{q}) + \varepsilon f_t(t, i, \mathbf{q}))| \leq r(t, \varepsilon; i, \mathbf{q}).$$

   - Taylor's theorem already implies $r(t, \varepsilon; i, \mathbf{q})/\varepsilon \to 0$.

2) $\frac{r(t,\varepsilon;i,\mathbf{q})}{\varepsilon} \downarrow 0$ as $\varepsilon \downarrow 0$.
3) $\int_0^\infty r(t, \varepsilon; i, \mathbf{q}) dt < \infty$, for ε small.
4) $f_t(\cdot)$ satisfies (4).

CONDITIONS

Non-exponential discounting:

- 1) and 4) reduce to

$$\delta \in \mathcal{C}_1 \quad \text{and} \quad \int_0^\infty \delta'(t)dt < \infty. \tag{8}$$

- 2) reduce to

$$\left| \frac{\delta(t+\varepsilon) - \delta(t)}{\varepsilon} - \delta'(t) \right| \quad \text{increasing in } \varepsilon, \quad \forall t \geq 0.$$

  This is ensured whenever δ is convex.

- 3) reduce to $\int_0^\infty |\delta(t+\varepsilon) - (\delta(t) + \varepsilon\delta'(t))|dt < \infty$. This is always true under (5) and (8).

Generalized hyperbolic (with exponent $k > 1$), *pseudo-exponential* discount functions satisfy these conditions.

Notation

- $F(Q) := (F(1, Q), F(2, Q), ..., F(N, Q))$.
- For any $i \in S$ and $Q \in \mathcal{Q}$, consider

$$G(i, Q) := \mathbb{E}_i \left[\int_0^\infty f_t(t, X_t, Q_{X_t}) dt \right].$$

Define

$$G(Q) := (G(1, Q), G(2, Q), ..., G(N, Q)).$$

The Expansion

For any $i \in S$ and $Q, Q^* \in \mathcal{Q}$, as $\varepsilon \downarrow 0$,

$$\begin{aligned} F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*) \\ &= \left(\Gamma^{Q^*}(Q_i^*) - \Gamma^{Q^*}(Q_i)\right)\varepsilon \\ &+ \frac{1}{2}\left(\Lambda^{Q^*}(i, Q^*) - \Lambda^{Q^*}(i, Q)\right)\varepsilon^2 + o(\varepsilon^2), \end{aligned} \tag{9}$$

where

$$\Gamma^{Q^*}(Q_i) := f(0, i, Q_i) + Q_i \cdot F(Q^*),$$
$$\Lambda^{Q^*}(i, Q) := f_t(0, i, Q_i) + Q_i \cdot \left(2G(Q^*) + \Gamma^{Q^*}(Q)\right).$$

- $\Gamma^{Q^*}(Q) = (\Gamma^{Q^*}(Q_1), \Gamma^{Q^*}(Q_2), ..., \Gamma^{Q^*}(Q_N))$.

Weak Equilibria

Theorem 1

$Q^* \in \mathcal{Q}$ is a **weak equilibrium** if and only if

$$\Gamma^{Q^*}(Q_i^*) \geq \Gamma^{Q^*}(Q_i) \quad \forall i \in S,\ Q \in \mathcal{Q}. \tag{10}$$

- Proof:

$$\frac{F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*)}{\varepsilon} = \left(\Gamma^{Q^*}(Q_i^*) - \Gamma^{Q^*}(Q_i)\right) + o(1),$$

which directly implies

$$\liminf_{\varepsilon \downarrow 0} \frac{F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*)}{\varepsilon} = \Gamma^{Q^*}(Q_i^*) - \Gamma^{Q^*}(Q_i).$$

CHARACTERIZATION

- (10) means: for any $i \in S$ and $Q \in \mathcal{Q}$,

$$f(0,i,Q_i^*) + Q_i^* \cdot F(Q^*) \geq f(0,i,Q_i) + Q_i \cdot F(Q^*). \tag{11}$$

  - (11) involves both Q^* and $Q \implies$ Hard to solve for Q^*.
  - **Idea:** Let Q approach Q^* in (11)
    $\implies$ get a *differential equation* involving Q^* only.

- Taking $Q_i = Q_i^* + \varepsilon\lambda \in D_i$ in (11) gives

$$f(0,i,Q_i^*) + Q_i^* \cdot F(Q^*) \geq f(0,i,Q_i^* + \varepsilon\lambda) + (Q_i^* + \varepsilon\lambda) \cdot F(Q^*).$$

$$\implies \frac{f(0,i,Q_i^* + \varepsilon\lambda) - f(0,i,Q_i^*)}{\varepsilon} + F(Q^*) \cdot \lambda \leq 0.$$

$$\implies \boxed{\Big(\nabla f(0,i,Q_i^*) + F(Q^*)\Big) \cdot \lambda \leq 0}.$$

Assume $\mathbf{q} \mapsto f(0, i, \mathbf{q})$ is $\mathcal{C}_1$, for all $i \in S$.

Proposition 1

Let $Q^* \in \mathcal{Q}$ be a **weak equilibrium**. For any $i \in S$ and $\lambda \in \mathfrak{T}$ s.t.

$$Q_i^* + \varepsilon\lambda \in D_i \quad \text{for } \varepsilon > 0 \text{ small enough,}$$

we have

$$\Big(\nabla f(0, i, Q_i^*) + F(Q^*)\Big) \cdot \lambda \leq 0.$$

- **Note:** Q^*, Q are generators of a Markov chain
  - For any $i \in S$, $\sum_{j=i}^{N} q_{ij}^* = 0$ and $\sum_{j=1}^{N} q_{ij} = 0$.
  - For any $i \in S$,

$$Q_i^* - Q_i \in \mathfrak{T} := \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1,\ldots,N} \lambda_i = 0 \right\}.$$

Corollary 1

Suppose $\mathbf{q} \mapsto f(0, i, \mathbf{q})$ is *concave*, for all $i \in S$.

Then, $Q^* \in \mathcal{Q}$ is a **weak equilibrium** if and only if

$$\Big(\nabla f(0, i, Q_i^*) + F(Q^*)\Big) \cdot \lambda \leq 0, \qquad (12)$$

for all $i \in S$ and $\lambda \in \mathfrak{T}$ s.t. $Q_i^* + \varepsilon\lambda \in D_i$ for $\varepsilon > 0$ small enough,

- **Proof:** Recall $\Gamma^{Q^*}(Q_i^*) = f(0, i, Q_i^*) + Q_i^* \cdot F(Q^*)$.
  - (12) $\implies$ Q_i^* is a local maximizer.
  - Concavity of f $\implies$ Q_i^* is a global maximizer.

That is, $\Gamma^{Q^*}(Q_i^*) \geq \Gamma^{Q^*}(Q_i)$ for all $Q \in \mathcal{Q}$.

- If Q_i^* is an interior point of D_i,

$$\text{for any } \lambda \in \mathfrak{T}, \quad Q_i^* + \varepsilon\lambda \in D_i \text{ for } \varepsilon > 0 \text{ small enough.}$$

  - Take $\lambda \in \mathfrak{T}$, with $\lambda_n = 1$, $\lambda_m = -1$, $\lambda_i = 0$ for $i \neq n, m$. Then $\Big(\nabla f(0, i, Q_i^*) + F(Q^*)\Big) \cdot \lambda \leq 0$ implies

$$\big(\partial_n f(0, i, Q_i^*) + F(n, Q^*)\big) - \big(\partial_m f(0, i, Q_i^*) + F(m, Q^*)\big) \leq 0$$

  - Take $\lambda \in \mathfrak{T}$, with $\lambda_n = -1$, $\lambda_m = 1$, $\lambda_i = 0$ for $i \neq n, m$. Then

$$-\big(\partial_n f(0, i, Q_i^*) + F(n, Q^*)\big) + \big(\partial_m f(0, i, Q_i^*) + F(m, Q^*)\big) \leq 0$$

Corollary 2

Let $Q^* \in \mathcal{Q}$ be a **weak equilibrium**. If Q_i^* is in the interior of D_i,

$$\partial_n f(0, i, Q_i^*) + F(n, Q^*) = \partial_m f(0, i, Q_i^*) + F(m, Q^*), \quad n, m = 1, \dots, N.$$

Strong Equilibria

Proposition 2

If $Q^* \in \mathcal{Q}$ satisfies

$$\Gamma^{Q^*}(Q_i^*) > \Gamma^{Q^*}(Q_i) \quad \forall i \in S \text{ and } Q \in \mathcal{Q} \text{ with } Q_i \neq Q_i^*,$$

then Q^* is a **strong equilibrium**.

- **Proof:** For any $Q \in \mathcal{Q}$ with $Q_i \neq Q_i^*$,

$$\Gamma^{Q^*}(Q_i^*) > \Gamma^{Q^*}(Q_i) \quad \text{and} \quad (9)$$

$$\implies \frac{F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*)}{\varepsilon} = \left(\Gamma^{Q^*}(Q_i^*) - \Gamma^{Q^*}(Q_i)\right) + o(1),$$

$$\implies F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*) > 0 \text{ as } \varepsilon > 0 \text{ small.}$$

- **Proof (conti.):**

  For any $Q \in \mathcal{Q} \setminus \{Q^*\}$ with $Q_i = Q_i^*$,

  - $q_{ij} = q_{ij}^* = 0$ for all $j \neq i$:

$$F(i, Q \otimes_\varepsilon Q^*) = \int_0^\infty f(t, i, Q_i) dt = F(i, Q^*) \quad \forall \varepsilon > 0.$$

  - $q_{ij} = q_{ij}^* > 0$ for all $j \neq i$: (9) reduces to

$$\frac{F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*)}{\varepsilon^2} = \frac{1}{2} Q_i^* \cdot \left(\Gamma^{Q^*}(Q^*) - \Gamma^{Q^*}(Q) \right) + o(1) = \underbrace{\sum_{j \neq i} q_{ij}^* \left(\Gamma^{Q^*}(Q_j^*) - \Gamma^{Q^*}(Q_j) \right)}_{> 0} + o(1).$$

$\Longrightarrow$ $F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*) > 0$ as $\varepsilon > 0$ small.

- **Proof (conti.):**

  For any $Q \in \mathcal{Q} \setminus \{Q^*\}$ with $Q_i = Q_i^*$,

  - $q_{ij} = q_{ij}^* > 0$ for some $j \neq i$: Consider

$$S_0 = \{j \in S : Q_j \neq Q_j^*\} \quad \text{and} \quad \tau := \inf\{t \geq 0 : X_t \in S_0\}.$$

Then

$$\begin{aligned} & F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*) \\ & = \mathbb{E}_i \left[\int_\tau^\infty f(t, X_t, Q_{X_t}) dt - \int_\tau^\infty f(t, X_t, (Q \otimes_\varepsilon Q^*)_{X_t}) dt \right] \\ & = \mathbb{E}_i \left[F(X_\tau, Q^*) - F(X_\tau, Q \otimes_{\varepsilon - \tau} Q^*) \mid \tau \leq \varepsilon \right] \mathbb{P}(\tau \leq \varepsilon) \\ & = \mathbb{E}_i \left[\underbrace{\left(\Gamma^{Q^*}(Q^*_{X_\tau}) - \Gamma^{Q^*}(Q_{X_\tau}) \right)}_{>0} (\varepsilon - \tau) \,\middle|\, \tau \leq \varepsilon \right] \cdot O(\varepsilon) \end{aligned}$$

$\Longrightarrow F(i, Q^*) - F(i, Q \otimes_\varepsilon Q^*) > 0$ as $\varepsilon > 0$ small.

A TWO-STATE MODEL

- $S = \{1, 2\}$.
- **Generator:** Any $Q \in \mathcal{Q}$ is of the form

$$Q = \begin{bmatrix} -a & a \\ b & -b \end{bmatrix}, \quad a, b \geq 0.$$

  Denote it by $Q \sim (a, b)$.
- **Pseudo-exponential discount function:**

$$\delta(t) = \frac{1}{2}\left(e^{-t} + e^{-2t}\right) \quad t \geq 0,$$

- **Payoff:**

$$f(t, 1, (-a, a)) = \delta(t) g_1(a) \quad \text{and} \quad f(t, 2, (b, -b)) = \delta(t) g_2(b),$$

  for some given functions g_1 and g_2.

A TWO-STATE MODEL

Let $Q \sim (a, b)$, $Q^* \sim (a^*, b^*)$ be given.

- **Notation:**

$$F(1, Q), F(2, Q) \implies F_1(a, b), F_2(a, b)$$
$$G(1, Q), G(2, Q) \implies G_1(a, b), G_2(a, b)$$
$$\Gamma^{Q^*}(Q_1), \Gamma^{Q^*}(Q_2) \implies \Gamma_1^{(a^*,b^*)}(a), \Gamma_2^{(a^*,b^*)}(b)$$
$$\Lambda^{Q^*}(1, Q), \Lambda^{Q^*}(2, Q) \implies \Lambda_1^{(a^*,b^*)}(a, b), \Lambda_2^{(a^*,b^*)}(a, b)$$

- **Explicit formulas:**

$$F_1(a, b) - F_2(a, b) = \frac{1}{2}\left(\frac{1}{1 + a + b} + \frac{1}{2 + a + b}\right)(g_1(a) - g_2(b)),$$
$$G_1(a, b) - G_2(a, b) = -\frac{1}{2}\left(\frac{1}{1 + a + b} + \frac{2}{2 + a + b}\right)(g_1(a) - g_2(b)).$$

EXAMPLE 1

Consider

$$g_1(a) = -a^2 \quad \text{and} \quad g_2(b) = 2 - (b-1)^2.$$

- By Corollaries 1 and 2, $Q \sim (a, b)$ is a weak equilibrium iff
  (i) if $a, b > 0$, we have

$$g_1'(a) + F_2(a,b) - F_1(a,b) = 0, \tag{13}$$
$$g_2'(b) + F_1(a,b) - F_2(a,b) = 0, \tag{14}$$

  (ii) if $a = 0$ (resp. $b = 0$), then "$\leq$" holds in (13) (resp. (14)).

$\Longrightarrow$ $Q^* \sim \left(\frac{5}{12}, \frac{7}{12}\right)$ is the *unique* **weak equilibrium**.

Example 1

- By Theorem 1, $a^* = \frac{5}{12}$, $b^* = \frac{7}{12}$ are maximizers of

$$\Gamma_1^{(a^*,b^*)}(a) = g_1(a) - a\left(F_1(a^*,b^*) - F_2(a^*,b^*)\right),$$
$$\Gamma_2^{(a^*,b^*)}(b) = g_2(b) + b\left(F_1(a^*,b^*) - F_2(a^*,b^*)\right).$$

  - Strict concavity of $g_1, g_2 \implies a^*, b^*$ are *strict* maximizers.
  - By Proposition 2, $Q^* \sim (\frac{5}{12}, \frac{7}{12})$ is a **strong equilibrium**.

EXAMPLE 2

Consider $g_1(a) = -a^2$ and

$$g_2(b) = \begin{cases} \frac{193}{144} + \frac{5}{6}b, & \text{for } b < \frac{7}{12}; \\ 2 - (b-1)^2, & \text{for } b \geq \frac{7}{12}. \end{cases}$$

First-order terms:

$$\Gamma_1^{(a^*,b^*)}(a) = -a^2 + (5/6)a,$$

$$\Gamma_2^{(a^*,b^*)}(b) = \begin{cases} \frac{193}{144}, & \text{if } b < \frac{7}{12}; \\ -\left(b - \frac{7}{12}\right)^2 + \frac{193}{144}, & \text{if } b \geq \frac{7}{12}. \end{cases},$$

- $\arg\max_{a \geq 0} \Gamma_1^{(a^*,b^*)}(a) = \{\frac{5}{12}\}$,
  $\arg\max_{b \geq 0} \Gamma_2^{(a^*,b^*)}(b) = [0, \frac{7}{12}]$.
- $Q^* \sim (\frac{5}{12}, \frac{7}{12})$ is a **weak equilibrium**.

EXAMPLE 2

Second-order term:

$$\Lambda_2^{(a^*,b^*)}(a^*,b) = -\frac{1}{12}b - \frac{579}{288}, \quad \text{for } b \le b^* = \frac{7}{12}.$$

- This shows that

$$\Lambda_2^{(a^*,b^*)}(a^*,b^*) < \Lambda_2^{(a^*,b^*)}(a^*,b), \quad \forall b \in [0,7/12).$$

- For any $Q \sim (a^*,b)$ with $b \in [0,7/12)$, (9) implies

$$F(2,Q^*) < F(2,Q \otimes_\varepsilon Q^*), \quad \text{for } \varepsilon > 0 \text{ small.}$$

- $Q^* \sim (\frac{5}{12}, \frac{7}{12})$ is *not* a **strong equilibrium**.

EXAMPLE 2

Question: Is there *any* strong equilibrium?

- Take $b = 0$ in (13) and (14) $\implies$

$$\frac{5}{6} \leq 2a = \frac{1}{2}\left(\frac{1}{1+a} + \frac{1}{2+a}\right)\left(a^2 + \frac{193}{144}\right).$$

- There is a unique solution $\bar{a} \geq 0$ ($\bar{a} \approx 0.42364$).
- First-order terms:

$$\Gamma_1^{(\bar{a},0)}(a) = -a(a - 2\bar{a}), \quad \Gamma_2^{(\bar{a},0)}(b) = \frac{193}{144} + (5/6 - 2\bar{a})\,b.$$

  - $a = \bar{a}$ is the unique maximizer of $\Gamma_1^{(\bar{a},0)}(a)$.
  - $b = 0$ is the unique maximizer of $\Gamma_2^{(\bar{a},0)}(b)$.
  - By Proposition 2, $Q = (\bar{a}, 0)$ is a **strong equilibrium**.

General Existence

Theorem

Suppose for any $i \in S$,

D_i *is a convex compact set and* $\mathbf{q} \mapsto f(0, i, \mathbf{q})$ *is concave.*

Then, there is a **weak equilibrium**.

- **Proof:** Define the set-valued map $\Phi : \mathcal{Q} \to 2^{\mathcal{Q}}$ by

$$\Phi(Q) := \left\{ R \in \mathcal{Q} : \; R_i \in \arg\max_{\mathbf{q} \in D_i} \left[f(0, i, \mathbf{q}) + \mathbf{q} \cdot F(Q) \right], \; \forall i \in S \right\}.$$

  - $\Phi(Q)$ is nonempty, closed, and convex, for all $Q \in \mathcal{Q}$.
  - Φ is upper semicontinuous (i.e. $R^n \to R$, $Q^n \to Q$, and $R^n \in \Phi(Q^n) \implies R \in \Phi(Q)$).

By Kakutani-Fan's theorem, $\exists Q^* \in \mathcal{Q}$ s.t. $Q^* \in \Phi(Q^*)$, i.e.

$$\Gamma^{Q^*}(Q^*) \geq \Gamma^{Q^*}(Q) \quad \forall Q \in \mathcal{Q}.$$

GENERAL EXISTENCE

Theorem

Suppose for any $i \in S$,

D_i *is a convex compact set and* $\mathbf{q} \mapsto f(0, i, \mathbf{q})$ *is strictly concave.*

Then, there is a **strong equilibrium**.

- **Proof:** Strict concavity of $\mathbf{q} \mapsto f(0, i, \mathbf{q})$ implies Q_i^* is the unique maximizer, i.e.

$$\Gamma^{Q^*}(Q^*) > \Gamma^{Q^*}(Q) \quad \forall Q \in \mathcal{Q},\ Q_i \neq Q_i^*.$$

By Proposition 2, Q^* is a strong equilibrium.

SUMMARY

- **New definition of continuous-time equilibria:**

  α^* is a strong equilibrium if for any (t, x) and α, there is $\varepsilon^*(t, x, \alpha) > 0$ such that

$$F(t, x, \alpha^*) \geq F(t, x, \alpha \otimes_{t+\varepsilon} \alpha^*), \quad \forall 0 < \varepsilon < \varepsilon^*.$$

- In a model with a continuous-time Markov chain,
  - Characterizations of strong and weak equilibria
  - Existence of strong and weak equilibria
  - Explicit demonstration of a **weak equilibrium that is not strong**.
- Future work: How about in a diffusion model?
  - He & Jiang (2018): weak, strong, regular equilibria.

THANK YOU!!

- *"Strong and Weak Equilibria for Time-Inconsistent Stochastic Control in Continuous Time"*
  (H. and Z. Zhou), available @ arXiv:1809.09243.