1.2.2 ▪ Elliptic-type PDEs

We restrict our discussion here to the case of Poisson's equation in 2-D

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f\,. \tag{1.15}$$

Equations in this "elliptic" category arise in numerous situations, such as for a *stream-function* in fluid mechanics or from *field equations* (describing gravitational and electrical fields, featuring *potentials* that satisfy *Laplace's equation*; equation (1.15) with RHS zero). Another source of equations of this type is equilibrium processes. For example, (1.15) arises in the $t\to\infty$ limit of the heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-f$.

Example 1. Create the following *compact* fourth-order accurate approximation for the 2-D Poisson's equation (a 2-D counterpart to (1.8)):

$$\begin{bmatrix} 1 & 4 & 1\\ 4 & -20 & 4\\ 1 & 4 & 1\end{bmatrix} u/(6h^2)=\begin{bmatrix} & 1 & \\ 1 & 8 & 1\\ & 1 & \end{bmatrix} f/12+O(h^4). \tag{1.16}$$

To derive this, we follow Collatz's *Mehrstellenverfahren* [49, 114]. Because of $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u=f$, it also holds that

$$\underbrace{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2}_{\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}} u=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f.$$

Approximation of these two relations to fourth and to second order, respectively, gives

$$\begin{bmatrix} & & -\frac{1}{12} & & \\ & & \frac{4}{3} & & \\ -\frac{1}{12} & \frac{4}{3} & -5 & \frac{4}{3} & -\frac{1}{12}\\ & & \frac{4}{3} & & \\ & & -\frac{1}{12} & & \end{bmatrix} u/h^2=[f]+O(h^4), \tag{1.17}$$

and

$$\begin{bmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1\\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix} u/h^4=\begin{bmatrix} & 1 & \\ 1 & -4 & 1\\ & 1 & \end{bmatrix} f/h^2+O(h^2), \tag{1.18}$$

respectively. Adding $\frac{1}{12}h^2$ times (1.18) to (1.17) eliminates the "outliers" and produces (1.16).

The formula (1.16) achieves its fourth order only thanks to the stencil for f in the right-hand side (RHS). As an approximation to the Laplace operator, the left-hand side (LHS) of (1.16) is accurate only to second order, as seen by Taylor expanding it around the center point:

$$\begin{aligned}
\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} u/(6h^2) = {} & \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u \\
& +\frac{1}{12}h^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 u \\
& +\frac{1}{360}h^4\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^4}{\partial x^4}+4\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)u \\
& +\frac{1}{60480}h^6\left(\frac{\partial^4}{\partial x^4}+4\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)\left(3\frac{\partial^4}{\partial x^4}+16\frac{\partial^4}{\partial x^2\partial y^2}+3\frac{\partial^4}{\partial y^4}\right)u \\
& +O(h^8).
\end{aligned}$$

For solutions to Laplace's equation, the first three RHS terms vanish, and the approximation becomes sixth-order accurate. The two key advantages of (1.16) over (1.17) are the following:

- The compact stencil is easier to use near boundaries.
- The diagonal dominance of coefficient matrix improves numerical stability and speeds up iterative solution methods.

Example 2. Analyze the accuracy of the hexagonal grid Laplace operator approximation (1.11).
Series expansion in the same style as in the previous example gives

$$\begin{aligned}
\begin{bmatrix} & 1 \quad 1 & \\ 1 & -6 & 1 \\ & 1 \quad 1 & \end{bmatrix} / \left(\tfrac{3}{2}h^2\right) = {} & \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u \\
& +\frac{1}{16}h^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 u \\
& +\frac{1}{5760}h^4\left(11\frac{\partial^6}{\partial x^6}+15\frac{\partial^6}{\partial x^4\partial y^2}+45\frac{\partial^6}{\partial x^2\partial y^4}+9\frac{\partial^6}{\partial y^6}\right)u \\
& +O(h^6).
\end{aligned}$$

This expansion confirms that the approximation is only second-order accurate for the Laplace operator but shows that it supports a compact fourth-order approximation for (1.15). However, the accuracy improves no further for solutions to Laplace's equation since the operator in the h^4-term does not factorize. It thus falls short in this regard of the Cartesian grid compact 9-point operator analyzed in Example 1.

Extending from 2-D to 3-D does not introduce any significant differences. For example, the 3-D counterpart to (1.16) becomes

$$\left[\begin{array}{ccccc} & & [0 & 1 & 0] \\ & [1 & 2 & 1] & \\ [0 & 1 & 0] & & \\ — & — & — & — & — \\ & & [1 & 2 & 1] \\ & [2 & -24 & 2] & \\ [1 & 2 & 1] & & \\ — & — & — & — & — \\ & & [0 & 1 & 0] \\ & [1 & 2 & 1] & \\ [0 & 1 & 0] & & \end{array}\right] u/(6h^2) = \left[\begin{array}{ccccc} & & [0 & 0 & 0] \\ & [0 & 1 & 0] & \\ [0 & 0 & 0] & & \\ — & — & — & — & — \\ & & [0 & 1 & 0] \\ & [1 & 6 & 1] & \\ [0 & 1 & 0] & & \\ — & — & — & — & — \\ & & [0 & 0 & 0] \\ & [0 & 1 & 0] & \\ [0 & 0 & 0] & & \end{array}\right] f/12 + O(h^4), \tag{1.19}$$

again combining fourth-order accuracy with diagonal dominance [305]. In this case, the 19-point stencil in the LHS is an $O(h^2)$ accurate approximation to the Laplacian operator, which reaches $O(h^4)$ for solutions to Laplace's equation—as it does for the Poisson's equation when used with the shown RHS stencil.

The last several examples have shown that FD approximations can provide higher orders of accuracy for PDEs than the orders by which they approximate individual derivative operators. This issue will come up again in the context of RBF-FD methods. Numerous generalizations of the compact formulas mentioned above have been described in the literature, including to variable coefficients, inclusion of lower-order terms, extensions to the coupled streamfunction-vorticity system for both steady and time-dependent 2-D Navier-Stokes equations, etc. [62, 175, 134, 176].