Optimal Equilibrium for Time Inconsistency

— the Stopping Case

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OUTLINE

Introduction

- What is time inconsistency?

Literature

- Two unsettled problems.

Methodology

- *Iterative approach*.
- Main results.

Examples

Classical Optimal Stopping

Consider

- a continuous Markovian process $X : [0, \infty) \times \Omega \mapsto \mathbb{R}^d$.
- a continuous payoff function $g : \mathbb{R}^d \mapsto \mathbb{R}_+$.

Optimal Stopping

Given $(t, x) \in [0, \infty) \times \mathbb{R}^d$, can we solve

$$\sup_{\tau \in \mathcal{T}_t} \mathbb{E}_{t,x}[\delta(\tau - t) g(X_\tau)]?$$

- $\mathcal{T}_t$: set of stopping times τ s.t. $\tau \geq t$ a.s.
- $\delta : \mathbb{R}_+ \mapsto [0, 1]$: decreasing from $\delta(0) = 1$.

Optimal Stopping Times [Karatzas & Shreve (1998)]

For all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the stopping time

$$\widetilde{\tau}(t,x) := \inf \Big\{ s \geq t : \delta(s-t) g(X_s^{t,x}) \\ = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_s} \mathbb{E}_{s, X_s^{t,x}} [\delta(\tau - t) g(X_\tau)] \Big\}$$

is optimal, i.e.

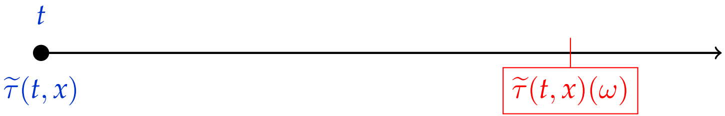
$$\mathbb{E}_{t,x}[\delta(\widetilde{\tau}(t,x) - t) g(X^{t,x}_{\widetilde{\tau}(t,x)})] = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_{t,x}[\delta(\tau - t) g(X_\tau)].$$

We say $\widetilde{\tau}$ is a **stopping policy**:

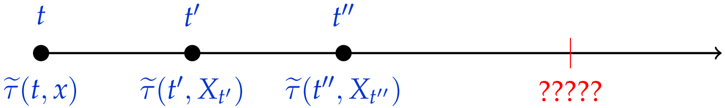
$$(t,x) \quad \longmapsto \quad \widetilde{\tau}(t,x) \in \mathcal{T}_t$$

Classical Theory: END OF STORY!

- **Problem Solved.** *Feeling Good?*



- **The Reality:**



- **Time Inconsistency:**
  - $\widetilde{\tau}(t, x)$, $\widetilde{\tau}(t', X_{t'})$, $\widetilde{\tau}(t'', X_{t''})$ may all be different.
  - Is it reasonable to apply $\widetilde{\tau}(t, x)$ at time t?

EXAMPLE (BES 1)

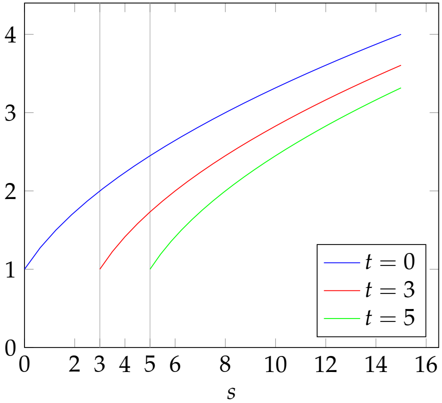
$$\sup_{\tau} \mathbb{E}_{t,x}\left[\frac{X_\tau}{1+(\tau-t)}\right].$$

- X_t : 1-D Bessel process
- Hyperbolic discount function $\delta(s)=\frac{1}{1+s}$, $s\geq 0$.
- By PDE approach, we solve explicitly:

$$\widetilde{\tau}(t,x)=\inf\left\{s\geq t : X_s^{t,x}\geq\sqrt{1+(s-t)}\right\}.$$

  - Free boundary $s\mapsto\sqrt{1+(s-t)}$ depends on initial time t.
  - This induces **time inconsistency**.

Free boundary $s \mapsto \sqrt{1+(s-t)}$ is changing over time t.



- $\widetilde{\tau}(t,x)$ not consistent over time.

EXAMPLE (SMOKING CESSATION)

- Smokers care most about:
  - long-term serious health problems
  - immediate pain from quitting smoking
- **Our Model:**
  - A smoker has a fixed lifetime T.
  - Deterministic cost process

$$X_s^{t,x} := xe^{\frac{1}{2}(s-t)}, \quad s \in [t, T]$$

  - Smoker can either
    - 1. quit at $s < T$ (costs X_s) 2. die peacefully at T (no cost)
    - 1. never quit (no cost) 2. die painfully at T (costs X_T)
  - Hyperbolic discounting:

$$\delta(s) = \frac{1}{1+s} \quad \forall s \geq 0.$$

- **Stopping problem:** For each $t \in [0, T]$,

$$\min_{s\in[t,T]} \delta(s-t)X_s^{t,x} = \min_{s\in[t,T]} \frac{xe^{\frac{1}{2}(s-t)}}{1+(s-t)}.$$

  - By Calculus, the optimal stopping time is

$$\widetilde{\tau}(t,x) = \begin{cases} t+1 & \text{if } t < T-1, \\ T & \text{if } t \geq T-1. \end{cases}$$

  - **time inconsistency $\implies$ procrastination**

Safe Case: Exponential Discounting

In classical literature of Mathematical Finance,

$$\delta(s) = e^{-\rho \cdot s} \quad \text{for some } \rho > 0.$$

- This means $\boxed{\delta(s-t)\delta(r-s) = \delta(r-t)}, \forall\, 0 \leq t \leq s \leq r.$
- Optimal stopping time becomes

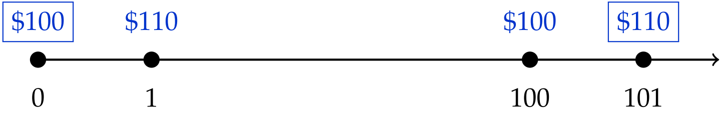
$$\widetilde{\tau}(t,x) := \inf \Big\{ s \geq t \;:\; \delta(s-t) g(X_s) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_s} \mathbb{E}_{s,X_s}[\delta(\tau - t) g(X_\tau)] \Big\}$$

$$= \inf \Big\{ s \geq t \;:\; g(X_s) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_s} \mathbb{E}_{s,X_s}[\delta(\tau - s) g(X_\tau)] \Big\}.$$

No t-dependence anymore!

Why not stay with exponential discounting?

- Payoff may not be monetary (utility, happiness, health,...).
- **Empirical:** people don't discount money exponentially.
  - People admit "decreasing impatience"
    (Laibson (1997), O'Donoghue & Rabin (1999))



$\frac{100}{\delta(1)200} > 1$ $\qquad$ $\frac{\delta(100)100}{\delta(101)200} < 1$

  - If $\delta(s-t) = e^{-\rho(s-t)}$,

$$\frac{100}{\delta(1)200} = \frac{\delta(100)100}{\delta(101)200} = \frac{e^{\rho}}{2} \text{ is constant.}$$

$\Rightarrow$ Does not capture "decreasing impatience".

Reasons for Time inconsistency:

- Non-exponential discounting (**our focus today**).
- Probability distortion:

$$\sup_{\tau \in \mathcal{T}_t} \int_0^\infty w\Big(\mathbb{P}_{t,x}\left[g(X_\tau) > u\right]\Big) du.$$

(H., Nguyen-Huu, X.Y. Zhou)

- Payoff depends on initial state (t, x):

$$\sup_{\tau \in \mathcal{T}_t} \mathbb{E}_{t,x}[g(t, x, X_\tau)].$$

- More general objective... (e.g. Mean-variance)

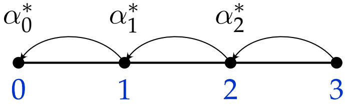
How to resolve time inconsistency? [Strotz (1955-56)]

1. Take into account future selves' behavior.
2. Find an *equilibrium* strategy that

once being enforced over time,
no future self would want to deviate from.

How to find an equilibrium?

- Backward sequential optimization [Pollak (1968)]:



- Limitation: infinite horizon? Continuous time?

Continuous-time Setup

- Ekeland & Lazrak (2006): provide precise mathematical definition of *equilibrium control* in continuous time.
- Ekeland & Pirvu (2008): Equilibria characterized by a system of nonlinear PDEs (*extended HJB equation*) –

$$\begin{cases} \textit{HJB-type equation} \\ \textit{equilibrium equation} \end{cases}$$

- Subsequent studies:
  Ekeland, Mbodji, & Pirvu (2012), Björk, Murgoci, & Zhou (2014), Dong & Sircar (2014), Björk & Murgoci (2014), Yong (2012),... (still many others).
- Almost all focused on control problems.

Two Problems

Unsettled challenging problems:

1) How to find *all* equilibria?
   - *Extended HJB* very difficult to solve: Wellposedness largely unestablished.
   - When solved in special cases, get *only one* equilibrium.
2) Which equilibrium to use?

This talk:

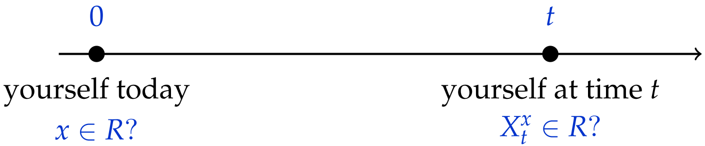
- Stopping problems.
- New method – *Iterative approach*.
- Resolve both 1) and 2).

From Now on...

- Assume X is *time-homogeneous*.

$$\mathbb{E}_{t,x}[\delta(\tau - t)g(X_\tau)] \implies \mathbb{E}_x[\delta(\tau)g(X_\tau)].$$

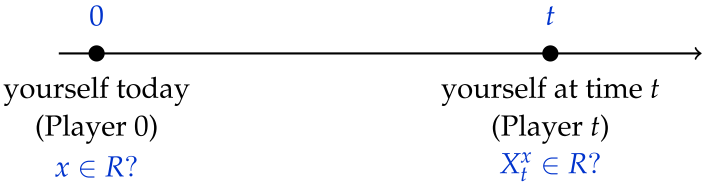
  - Mainly for simplicity of presentation.
- Given $R \in \mathcal{B}(\mathbb{R}^d)$,



  - Will always refer to the *stopping criterion* R, instead of τ.
  - Markovian stopping strategy is consistent with classical optimal stopping.

Game-Theoretic Approach

- Given a *stopping criterion* $R \in \mathcal{B}(\mathbb{R}^d)$,



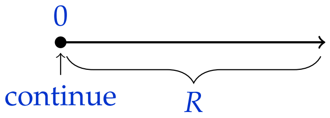
- **Game-theoretic** thinking of Player 0:
  Given that each Player t will follow R,
  - what is the best stopping strategy at time 0?
  - can it just be R?

Best Stopping Strategy

Player 0 has only **two** possible actions: to stop or to continue.

- If she stops, gets $g(x)$ right away.
- If she continues, she will eventually stop at the moment

$$\rho(x,R) := \inf\{t > 0 \;:\; X_t^x \in R\}.$$



- **Note:** Need $t > 0$ (not $t \geq 0$):

  "$t > 0$" means "*continuation at time* 0*, regardless of* R".

- Her expected payoff is then $\mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})]$.

The best stopping strategy for Player 0:

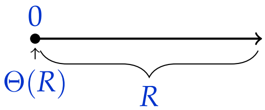
- $g(x) > \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})] \Rightarrow$ **stop** at time 0
- $g(x) < \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})] \Rightarrow$ **continue** at time 0
- $g(x) = \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})] \Rightarrow$
  - **indifferent** between to stop and to continue at time 0.
  - no incentive to deviate from R.

- Summarize the best stopping strategy for Player 0 as

$$\Theta(R) := S_R \cup (I_R \cap R),$$

where

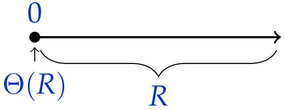
$$S_R := \{x : g(x) > \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})]\},$$
$$I_R := \{x : g(x) = \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})]\},$$
$$C_R := \{x : g(x) < \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})]\}.$$



- In general, $\Theta(R) \neq R$.
  - Player 0 wants to follow $\Theta(R)$, instead of R.

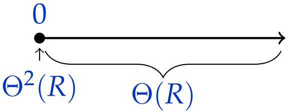
IMPROVING VIA ITERATION

1. At first, one follows $R \in \mathcal{B}(\mathbb{R}^d)$. By game-theoretic thinking,



$\Longrightarrow$ switch from R to $\Theta(R)$

2. Now, one follows $\Theta(R)$. By game-theoretic thinking,



$\Longrightarrow$ switch from $\Theta(R)$ to $\Theta^2(R)$

3. Continue this procedure *until* we reach

$$R_0 := \lim_{n\to\infty} \Theta^n(R)$$

Expect: $\Theta(R_0) = R_0$, i.e. cannot improve anymore.

EQUILIBRIUM

Definition

$R \in \mathcal{B}(\mathbb{R}^d)$ is called an **equilibrium** if $\Theta(R) = R$.

- **Trivial Equilibrium:** consider $R := \mathbb{R}^d$.

$$\rho(x,R) = \inf\{t > 0 \ : \ X_t^x \in R\} = 0$$
$$\implies \quad \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})] = g(x) \quad \implies \quad x \in I_R.$$

Thus $I_R = \mathbb{R}^d$, so $\Theta(R) = S_R \cup (I_R \cap R) = R$.

- **In general**, given any $R \in \mathcal{B}(\mathbb{R}^d)$, carry out iteration:

$$R \longrightarrow \Theta(R) \longrightarrow \Theta^2(R) \longrightarrow \cdots \longrightarrow \text{“equilibrium”??}$$

- **To show:**
(i) $R_0 := \lim_{n\to\infty} \Theta^n(R)$ converges (ii) $\Theta(R_0) = R_0$.

Decreasing Impatience

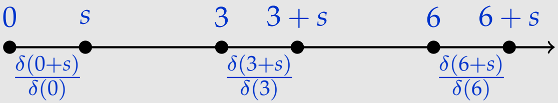
- **Assumption:** the discount function $\delta : \mathbb{R}_+ \mapsto [0,1]$ satisfies

$$\delta(t)\delta(s) \leq \delta(t+s) \quad \forall\, t,s \geq 0. \tag{1}$$

Definition

A discount function δ induces **Decreasing Impatience** if,

for any $s > 0$, $\frac{\delta(t+s)}{\delta(t)}$ is increasing in t.



$$\text{DI} \implies \frac{\delta(t+s)}{\delta(t)} \geq \frac{\delta(0+s)}{\delta(0)} = \delta(s) \implies \delta(t)\delta(s) \leq \delta(t+s).$$

- Once we consider **DI**, (1) is automatically satisfied.

Main Result I

Lemma

Assume (1). *For any* $R \in \mathcal{B}(\mathbb{R}^d)$,

$$\text{if} \quad \boxed{R \subseteq \Theta(R)}, \quad \text{then} \quad \Theta^n(R) \subseteq \Theta^{n+1}(R) \ \ \forall n.$$

Theorem

Assume (1) *and* $\boxed{R \subseteq \Theta(R)}$. *Then,*

$$R_0 := \lim_{n\to\infty} \Theta^n(R) = \bigcup_{n\in\mathbb{N}} \Theta^n(R).$$

Moreover, R_0 *is an equilibrium, i.e.* $\Theta(R_0) = R_0$.

$\boxed{R \subseteq \Theta(R)}$ is satisfied whenever R is open.

- For any $x \in R$, $x \in B_\varepsilon(x) \subseteq R$ for some $\varepsilon > 0$. Then,

$$\rho(x, R) = \inf\{t > 0 \;:\; X_t^x \in R\} = 0.$$

  This implies

$$\mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})] = g(x), \quad \text{i.e. } x \in I_R.$$

  Thus, $R \subseteq I_R$.
- $\Theta(R) = S_R \cup (I_R \cap R) = S_R \cup R \supseteq R.$

EXAMPLE (SMOKING CESSATION)

Classical Optimal Stopping:

- For each $t \in [0, T]$,

$$\min_{s\in[t,T]} \delta(s-t)X_s^{t,x} = \min_{s\in[t,T]} \frac{xe^{\frac{1}{2}(s-t)}}{1+(s-t)}.$$

- The optimal stopping time is

$$\widetilde{\tau}(t,x) = \begin{cases} t+1 & \text{if } t < T-1, \\ T & \text{if } t \geq T-1. \end{cases}$$ (**procrastination**)

- This corresponds to the *stopping criterion*:

$$\widetilde{R} := \{T\} \times \mathbb{R}_+.$$

Iterative Approach:

- Find the equilibrium $\widetilde{R}_0 := \lim_{n\to\infty} \Theta^n(\widetilde{R})$.
  - First iteration: $\Theta(\widetilde{R}) = S_{\widetilde{R}} \cup (I_{\widetilde{R}} \cap \widetilde{R})$, where

$$S_{\widetilde{R}} := \{(t,x) : x < \delta(\rho(t,x,\widetilde{R}) - t)X_{\rho(t,x,\widetilde{R})}\},$$
$$I_{\widetilde{R}} := \{(t,x) : x = \delta(\rho(t,x,\widetilde{R}) - t)X_{\rho(t,x,\widetilde{R})}\}.$$

  - $\Theta(\widetilde{R}) = ([0, T - s^*] \cup \{T\}) \times \mathbb{R}_+$, with $s^* \approx 2.513$.
  - Second iteration: $\Theta^2(\widetilde{R}) = \Theta(\widetilde{R})$. Thus, $\Theta(\widetilde{R}) \in \mathcal{E}$.
- **Conclude:**

$$\widetilde{R}_0 := \lim_{n\to\infty} \Theta^n(\widetilde{R}) = \Theta(\widetilde{R}) = ([0, T - s^*] \cup \{T\}) \times \mathbb{R}_+.$$

$\widetilde{R}_0$ says "Stop Smoking Immediately!!"

(unless you're too old...)

COMPARISON WITH STANDARD APPROACH

How to find all equilibria?

- Standard Approach:

  Solve extended HJB equation, a system of nonlinear PDEs

  - Difficult to solve....
  - Even when it is solved, just obtain *one* equilibrium

  $\Longrightarrow$ *How to find other equilibria?*

- Iterative Approach:

  Do fixed-point iterations

  - Easy to implement
  - Easy to find different equilibriums...
    - just by changing the initial strategy R.
    - In some cases, we can find *all* equilibria.

Example (BES1)

$$\sup_{\tau \in \mathcal{T}_t} \mathbb{E}_{t,x}\left[\frac{X_\tau}{1+(\tau - t)}\right] \implies \sup_{\tau \in \mathcal{T}} \mathbb{E}_{x}\left[\frac{X_\tau}{1+\tau}\right]$$

- X_t : 1-D Bessel process.
- Can characterize the <u>whole</u> set $\mathcal{E}$ of equilibria:

$$\mathcal{E} = \{[a, \infty) : a \in [0, a^*]\},$$

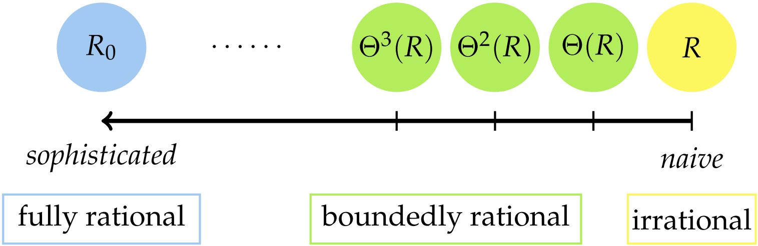
where a^* solves

$$a \int_0^\infty e^{-s}\sqrt{2s}\tanh(a\sqrt{2s})ds = 1 \quad (\implies a^* \approx 0.946).$$

Our Iterative Approach:

$$R_0 = \lim_{n \to \infty} \Theta^n(R)$$

embodies the *hierarchy of strategic reasoning* (Stahl (1993)):



- **Bounded Rationality** proposed by H. Simon (1982).
- **Which equilibrium to use?** It's agent-based...

Which equilibrium should we employ?

- Try to find an "optimal" equilibrium:

$$\sup_{R \in \mathcal{E}} \mathbb{E}_x[\delta(\rho(x,R))g(X_{\rho(x,R)})].$$

- Major concern:
  The optimal equilibrium may depend on x
  $\implies$ generates a new level of time inconsistency...

- Under "$\delta(t)\delta(s) \leq \delta(t+s)$",
  an optimal equilibrium exists!
  - It generates larger value than any other equilibrium does, at any state x!

Definition

For any $R \in \mathcal{E}$, define

$$V(x, R) := g(x) \vee J(x, R) \quad \text{for all } x,$$

where

$$J(x, R) = \mathbb{E}_x[\delta(\rho(x, R))g(X_{\rho(x,R)})].$$

Definition

$R^* \in \mathcal{E}$ is called an **optimal equilibrium** if, for any $R \in \mathcal{E}$,

$$V(x, R^*) \geq V(x, R) \quad \text{for all } x.$$

The Smaller, the Better

- Assume $\delta(t)\delta(s) \leq \delta(t+s), \forall s, t > 0$.
- For any $\boxed{R \subseteq T}$ with $R \in \mathcal{E}$, let

$$\tau_R^x := \rho(x, R), \quad \tau_T^x := \rho(x, T), \quad A := \{\tau_T^x < \tau_R^x\}.$$

$$\begin{aligned}
J(x, R) &= \mathbb{E}_x[\delta(\tau_R^x) g(X_{\tau_R^x})] \\
&= \mathbb{E}_x[\delta(\tau_R^x) g(X_{\tau_R^x}) 1_A] + \mathbb{E}_x[\delta(\tau_R^x) g(X_{\tau_R^x}) 1_{A^c}] \\
&= \mathbb{E}_x\left[\mathbb{E}_x[\delta(\tau_R^x) g(X_{\tau_R^x}) \mid \mathcal{F}_{\tau_T^x}] 1_A\right] + \mathbb{E}_x[\delta(\tau_T^x) g(X_{\tau_T^x}) 1_{A^c}] \\
&\geq \mathbb{E}_x\left[\delta(\tau_T^x) \mathbb{E}_x[\delta(\tau_R^x - \tau_T^x) g(X_{\tau_R^x}) \mid \mathcal{F}_{\tau_T^x}] 1_A\right] + \mathbb{E}_x[\delta(\tau_T^x) g(X_{\tau_T^x}) 1_{A^c}] \\
&= \mathbb{E}_x[\delta(\tau_T^x) J(X_{\tau_T^x}, R) 1_A] + \mathbb{E}_x[\delta(\tau_T^x) g(X_{\tau_T^x}) 1_{A^c}] \\
&\geq \mathbb{E}_x[\delta(\tau_T^x) g(X_{\tau_T^x}) 1_A] + \mathbb{E}_x[\delta(\tau_T^x) g(X_{\tau_T^x}) 1_{A^c}] = J(x, T).
\end{aligned}$$

- $R, T \in \mathcal{E}$ with $R \subseteq T \implies V(x, R) \geq V(x, T)$.

The Smaller, the Better

- Take arbitrary $R, T \in \mathcal{E}$.

$$R \cap T \subseteq R, \quad R \cap T \subseteq T.$$

- If $R \cap T \in \mathcal{E}$,
  - previous argument implies

$$V(x, R \cap T) \geq V(x, R), \quad V(x, R \cap T) \geq V(x, T).$$

  - **Guess:** *"optimal equilibrium = intersection of all equilibria"*.
- Does "$R \cap T \in \mathcal{E}$, for $R, T \in \mathcal{E}$" hold?
  - In general, not so sure....
  - Can be established in one-dimensional case.

One-Dimensional Case

Assumption

For any $x \in \mathbb{R}$,

$$\mathbb{P}[\overline{X}_t^x > x] = \mathbb{P}[\underline{X}_t^x < x] = 1 \quad \forall t > 0, \tag{2}$$

where $\overline{X}_t^x := \max_{s\in[0,t]} X_s^x$ and $\underline{X}_t^x := \min_{s\in[0,t]} X_s^x$

- $\implies$ If X reaches ∂R, enters R immediately.
- Let X be given by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

with $\sigma(\cdot) > 0$. Define $\theta(\cdot) := \mu(\cdot)/\sigma(\cdot)$. If the process

$$Z_t := \exp\left(-\int_0^t \theta(X_s)dB_s - \frac{1}{2}\int_0^t \theta^2(X_s)ds\right) \quad t \geq 0$$

is a martingale, then (2) is satisfied.

Main Result II

Lemma

Assume $\delta(t)\delta(s) \leq \delta(t+s)$ and (2). Then,

$$R, T \in \mathcal{E} \implies R \cap T \in \mathcal{E}.$$

Theorem

Assume $\delta(t)\delta(s) \leq \delta(t+s)$ and (2). Then,

$$R^* := \bigcap_{R \in \mathcal{E},\ R \text{ closed}} R$$

is an *optimal equilibrium*.

EXAMPLE (BES1)

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\frac{X_\tau}{1+\tau} \right]$$

- X_t : 1-D Bessel process.
- Can characterize the whole set $\mathcal{E}$ of equilibria:

$$\mathcal{E} = \{[a, \infty) : a \in [0, a^*]\},$$

where a^* solves

$$a \int_0^\infty e^{-s}\sqrt{2s} \tanh(a\sqrt{2s})ds = 1 \quad (\implies a^* \approx 0.946).$$

- Optimal equilibrium:

$$R^* = \bigcap_{a \in [0, a^*]} [a, \infty) = [a^*, \infty).$$

Example (a Put on GBM)

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[\frac{(K-X_\tau)^+}{1+\tau}\right]$$

- X_t is GBM: $\quad dX_t = \mu X_t dt + \sigma X_t dW_t.$
- Can prove that
  - (i) $(0, (\frac{\lambda}{1+\lambda})K]$ is an equilibrium, where

$$\lambda := \int_0^\infty e^{-s}\left(\sqrt{(\mu/\sigma^2-1/2)^2+2s/\sigma^2}+(\mu/\sigma^2-1/2)\right)ds > 0.$$

  - (ii) every closed equilibrium must contain $(0, (\frac{\lambda}{1+\lambda})K]$.
- Optimal equilibrium:

$$R^* = \bigcap_{R\in\mathcal{E},\ R\ \text{closed}} R = \left(0, \frac{\lambda K}{1+\lambda}\right].$$

Summary

How to find all equilibria?

- Do fixed-point iterations.
  - Easy to implement.
  - Easy to find many equilibria, and possibly *all* equilibria.

Which equilibrium to use?

- Introduce *optimal equilibrium*: it generates larger value than any other equilibrium, at any state x.
- In 1-D, there exists an *optimal equilibrium*, taking the form

$$R^* = \bigcap_{R \in \mathcal{E},\ R \text{ closed}} R.$$

THANK YOU!!

- *"Time-consistent stopping under decreasing impatience"*
  (H. and Nguyen-Huu), to appear in Finance & Stochastics.
- *"Stopping Behaviors of Naive and Non-Committed Sophisticated Agents when They Distort Probability"*
  (H., Nguyen-Huu, and X.Y. Zhou), Available @ arXiv:1709.03535.
- *"Optimal Equilibrium for Time-Inconsistent Stopping Problems – the Discrete-Time Case"*
  (H. and Z. Zhou), available @ arXiv:1707.04981.
- *"Optimal Equilibrium for Time-Inconsistent Stopping Problems – the Continuous-Time Case"*
  (H. and Z. Zhou), first draft in preparation.