The Stochastic Solution to a Cauchy Problem Associated with Nonnegative Price Processes

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Under no-arbitrage pricing, one typically postulates a diffusion model for the stock X under some risk-neutral measure:

$$dX_s^{t,x} = \sigma(X_s^{t,x})dW_s,\ X_t^{t,x} = x \geq 0, \tag{1}$$

with

$$\sigma(x) = 0 \text{ for } x \leq 0; \text{ and } \sigma(x) > 0 \text{ for } x > 0. \tag{2}$$

This in particular captures the phenomenon of **bankruptcy**.

For a payoff function $g : [0, \infty) \mapsto \mathbb{R}$, the value function of a European contingent claim is given by

$$U(t, x) := \mathbb{E}^{t,x}[g(X_T^{t,x})]. \tag{3}$$

By a heuristic use of Itô's rule, U should **potentially** be a classical solution to the Cauchy problem

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2(x)\partial_{xx} u = 0, & (t,x) \in [0,T) \times (0,\infty); \\ u(T,x) = g(x), & x \in (0,\infty); \\ u(t,0) = g(0), & t \in [0,T]. \end{cases} \tag{4}$$

Difficult to verify the smoothness of U, as standard results of parabolic equations (e.g. Lieberman (1996)) cannot be applied, mainly due to

- the **degeneracy** of (4) at the boundary $x = 0$.
- the lack of suitable **growth condition** on σ.

Observe: f is smooth and satisfies (4)

$\Rightarrow f(t \wedge T, X_{t \wedge T})$ is a local martingale.

DEFINITION [BAYRAKTAR & SÎRBU (2012)]

A measurable function $u : [0, T] \times [0, \infty) \mapsto \mathbb{R}$ is said to be a (local) stochastic solution to (4) if

(I) for any $(t, x) \in [0, T] \times \mathbb{R}$ and any weak solution $(X^{t,x}, W^{t,x}, \Omega^{t,x}, \mathcal{F}^{t,x}, \mathbb{P}^{t,x}, \{\mathcal{F}^{t,x}_s\}_{s \geq t})$ of (1),

$$u\left(r \wedge T, X^{t,x}_{r \wedge T}\right) \text{ is a } \mathbb{P}^{t,x}\text{-(local) martingale.}$$

(II) $u(T, x) = g(x)$ for $x \in (0, \infty)$, $u(t, 0) = g(0)$ for $t \in [0, T]$.

The notion of stochastic solutions was introduced in Stroock & Varadhan (1972). Recent developments: Janson & Tysk (2006), Ekström & Tysk (2009), Bayraktar, Kardaras, & Xing (2012).

local stochastic solution $+$ "continuity" $=$ classical solution

- A classical solution is a **local** stochastic solution.
- There can be at most one stochastic solution.
- Assume
  1. g **is of linear growth**;
  2. **uniqueness in law of weak solutions to** (1) **holds**, i.e.

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{\sigma^2(y)} dy < \infty \text{ for some } \varepsilon > 0, \quad \forall\, x > 0. \tag{5}$$

Then $U(t,x) = \mathbb{E}^{t,x}[g(X_T^{t,x})]$ is the stochastic solution to (4).

Standing Assumption:

1. g is of linear growth.
2. uniqueness of weak solutions holds for (1).

Goal: Analyze various properties of the stochastic solution U.
More specifically,

- Under what condition can we characterize U as the unique local stochastic solution?
- What condition guarantees that U is smooth? And when can U be characterized as the unique classical solution to (4)?
- If U may not be smooth, how should we characterize U?

First, we focus on the special case where $g(x) \equiv x$. The Cauchy problem (4) reduces to

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2(x)\partial_{xx}u = 0, & (t,x) \in [0,T) \times (0,\infty); \\ u(T,x) = x, & x \in (0,\infty); \\ u(t,0) = 0, & t \in [0,T]. \end{cases} \tag{4'}$$

Consider $\hat{D} := \{u : \exists K > 0 \text{ s.t. } |u(t,x)| \leq K(1+x)\ \forall\ (t,x)\}$.

Lemma

"(4') admits a unique local stochastic solution in $\hat{D}$"

$\Rightarrow$ "X is a martingale"

Proof: Suppose X is a strict local martingale. Then $U(t,x) = \mathbb{E}^{t,x}[X_T^{t,x}] < x$ and $\tilde{U}(t,x) := x$ are two distinct local stochastic solutions to (4') in $\hat{D}$, a contradiction.

Lemma

"X is a martingale"
$\Rightarrow$ "(4') admits a unique local stochastic solution in $\hat{D}$"

Sketch of proof: For any local stochastic solution u to (4') in $\hat{D}$, consider

$$\tau^{\beta} := \inf\{s \geq t : X_s^{t,x} \geq \beta\} \wedge T. \tag{6}$$

1. By **optional sampling** (!?),
"$u(s \wedge T, X_{s\wedge T}^{t,x})$ is a local martingale and $\tau^{\beta} \leq T$"
$\Rightarrow$ "$u(s \wedge \tau^{\beta}, X_{s\wedge \tau^{\beta}}^{t,x})$ is a local martingale"

Note that $u(s \wedge \tau^{\beta}, X_{s\wedge \tau^{\beta}}^{t,x})$ is actually bounded, and therefore must be a true martingale. Hence,

$$u(t,x) = \mathbb{E}^{t,x}[u(T \wedge \tau^{\beta}, X_{T\wedge \tau^{\beta}}^{t,x})], \quad \forall \beta > 0.$$

2. It follows that

$$\begin{aligned} u(t,x) &= \lim_{\beta\to\infty} \mathbb{E}^{t,x}[u(T\wedge\tau^\beta, X^{t,x}_{T\wedge\tau^\beta})] \\ &= \lim_{\beta\to\infty} \mathbb{E}^{t,x}[u(T, X^{t,x}_T)1_{\{\tau^\beta=T\}}] + \lim_{\beta\to\infty} \mathbb{E}^{t,x}[u(\tau^\beta,\beta)1_{\{\tau^\beta<T\}}] \\ &= \lim_{\beta\to\infty} \mathbb{E}^{t,x}[X^{t,x}_T 1_{\{\tau^\beta=T\}}] + \lim_{\beta\to\infty} \mathbb{E}^{t,x}[u(\tau^\beta,\beta)1_{\{\tau^\beta<T\}}] \\ &= U(t,x) + \lim_{\beta\to\infty} \mathbb{E}^{t,x}[u(\tau^\beta,\beta)1_{\{\tau^\beta<T\}}]. \end{aligned}$$

3. $\mathbb{E}^{t,x}[u(\tau^\beta,\beta)1_{\{\tau^\beta<T\}}] \to 0$?

Since $u \in \hat{D}$, $|\mathbb{E}^{t,x}[u(\tau^\beta,\beta)1_{\{\tau^\beta<T\}}]| \le K(1+\beta)\mathbb{P}(\tau^\beta < T)$. Also,

$$\mathbb{E}[X^{t,x}_{\tau^\beta}] = \mathbb{E}[\beta 1_{\{\tau^\beta<T\}}] + \mathbb{E}[X^{t,x}_T 1_{\{\tau^\beta=T\}}] \to \lim_{\beta\to\infty} \beta\mathbb{P}(\tau^\beta < T) + \mathbb{E}[X^{t,x}_T].$$

Since X is a martingale, conclude

$$\lim_{\beta\to\infty} \beta\mathbb{P}(\tau^\beta < T) = 0.$$

A technical detail:
In **Step 1** above, "optional sampling" is actually NOT applicable to $u(s \wedge T, X^{t,x}_{s \wedge T})$, as the process may not be right continuous (note that we do not assume any continuity on u).

We need to

- localize $u(s \wedge T, X^{t,x}_{s \wedge T})$ (this gives **martingality**).
- work with the right continuous modifications of these localized processes. (this give **right-continuity**).

MAIN RESULT I

The Following are equivalent:

(I) (4) admits a unique local stochastic solution in $\hat{D}$, $\forall g \in \hat{D}$.

(II) (4') admits a unique local stochastic solution in $\hat{D}$ $(g(x) \equiv x)$.

(III) X is a martingale.

(IV) $\int_1^\infty \frac{x}{\sigma^2(x)} dx = \infty$.

Note: (iii)$\Leftrightarrow$(iv) was identified in Delbaen & Shirakawa (2002), assuming that σ is locally bounded. We generalize their result to the case where σ satisfies the local integrability condition (5).

A Consequence of Main Result I

Recall that {classical solutions} $\subseteq$ {local stochastic solutions}.

Feynman-Kac formula

Suppose $\int_1^\infty \frac{x}{\sigma^2(x)}dx = \infty$. Then, if $u \in \hat{D}$ is a classical solution to (4), then u admits the stochastic representation
$u(t,x) = \mathbb{E}^{t,x}[g(X_T^{t,x})]$ $(= U)$.

Remark:

- Standard Feynman-Kac formula requires continuity and linear growth condition on σ (see Friedman (2006), Karatzas & Shreve (1991)). Here, no continuity on σ is assumed, and $\int_1^\infty \frac{x}{\sigma^2(x)}dx = \infty$ is weaker than linear growth condition.
- This generalizes Theorem 1 in Bayraktar & Xing (2010) to the case without **local boundedness** of σ.

Connection to Classical Solutions

Question

When is $U(t,x) := E^{t,x}[g(X_T^{t,x})]$ a classical solution?

classical solution

$$= \text{interior smoothness} + \text{continuity to boundary}$$

Weakest condition in literature: σ is locally 1/2-Hölder continuous (Ekström & Tysk (2009)).

Standard methodology: Construct a monotone smooth approximation for U, by using

local stochastic solution + "continuity" = classical solution

- interior smoothness of U: obtained by **Schauder estimates**.
- By exploiting the **monotonicity** of the approximation, continuity of U up to the boundary is also obtained

Connection to Classical Solutions

Claim: σ being locally δ-Hölder continuous (with $\delta > 0$) is enough.

- Classical PDE literature: δ-Hölder continuous (with $\delta > 0$) plays a crucial role in constructing a smooth solution to a parabolic equation. **Yet, whether** $\delta \geq 1/2$ **does not matter**.
- locally $1/2$-Hölder continuous is the minimal condition which guarantees the **existence of a unique strong solution to** (1). This facilitates deriving *a priori* continuity of U.

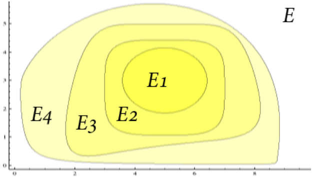
Assume only locally δ-Hölder continuous (with $\delta > 0$) on σ

$\Rightarrow$ construction in Ekström & Tysk (2009) fails...

$\Rightarrow$ New methodology is needed!

Assume: σ is locally δ-Hölder continuous (with $\delta > 0$).

1. Take g_n continuous s.t. $g_n \to g$. Take an increasing sequence $\{E_n\}_{n\in\mathbb{N}}$ of compact subsets of $E := [0, T] \times [0, \infty)$.



On each E_n, by classical PDE results (see e.g. Lieberman (1996)), can construct a classical solution $u_n \in H^{2+\delta}(E_n)$ to

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2 \partial_{xx} u = 0 & \text{in } E_n, \\ u(t,x) = g_n(x) & \text{on } \partial^* E_n. \end{cases} \qquad (\text{PDE}_n)$$

Moreover, the Hölder constant depends on only E_n.

Deriving Interior Smoothness of U

2. By **Arzela-Ascoli-type argument**, there exists $\hat{u}$ s.t.

$$\text{for each } n \in \mathbb{N},\ \{u_k\}_{k\geq n} \text{ converges to } \hat{u} \text{ in } H^{2+\delta}(E_n).$$

This in particular implies that $\hat{u} \in C^{1,2}([0,T) \times (0,\infty))$.

3. Since u_n is a smooth solution to (PDE$_n$), by Itô's rule

$$u_n(t,x) = \mathbb{E}^{t,x}[g_n(X^{t,x}_{\tau^n})] \quad \text{for } (t,x) \in E_n,$$

where $\tau^n := \inf\{s \geq t : (s, X^{x,t}_s) \notin E_n\} \leq T$. Then

$$\hat{u}(t,x) = \lim_{n\to\infty} u_n(t,x) = \lim_{n\to\infty} \mathbb{E}^{t,x}[g_n(X^{t,x}_{\tau^n})] = \mathbb{E}^{t,x}[g(X^{t,x}_T)]$$
$$= U(t,x).$$

Interior Smoothness

Suppose σ **is locally Hölder continuous with exponent** $\delta \in (0,1]$. Then, for any nonnegative continuous $g \in \hat{D}$, the stochastic solution U belongs to $C^{1,2}([0,T) \times (0,\infty))$ and solves (4).

What about continuity of U up to the boundary??

- We will NOT rely on any smooth approximation.
- We will use the techniques of viscosity solutions developed in Bayraktar & Sîrbu (2012).

Review of Bayraktar & Sîrbu (2012):

- We say a measurable function $u : [0, T] \times [0, \infty) \mapsto \mathbb{R}$ is a **stochastic subsolution** to (4) if
  - (I) For any weak solution to (1) with initial condition (t, x),

$$u\left(r \wedge T, X^{t,x}_{r\wedge T}\right) \text{ is a submartingale.}$$

  - (II) $u(T, x) \leq g(x)$ for $x \in (0, \infty)$, $u(t, 0) \leq g(0)$ for $t \in (0, T]$.

Review of Bayraktar & Sîrbu (2012) [conti.]:

- $\mathcal{U}_g^- := \{$LSC stochastic subsolutions to (4)$\}$.
- Suppose $\mathcal{U}_g^- \neq \emptyset$. Given $u \in \mathcal{U}_g^-$, $u(t,x) \leq \mathbb{E}^{t,x}[g(X_T^{t,x})] = U(t,x)$. It follows that

$$v_g^-(t,x) := \sup_{u \in \mathcal{U}_g^-} u(t,x) \leq U(t,x).$$

By definition, v_g^- is LSC. Moreover, if g is LSC, then

$$v_g^-(t,x) \text{ is a viscosity supersolution to (4),}$$
$$v_g^-(T,x) = g(x)$$

CONTINUITY UP TO THE BOUNDARY

Let $g \in \hat{D}$ be nonnegative and continuous. Then, the stochastic solution U satisfies the following:

$$\begin{aligned} U^*(T,x) = U_*(T,x) = g(x) \ \text{ for } x \in (0,\infty), \\ U^*(t,0) = U_*(t,0) = g(0) \ \text{ for } t \in [0,T]. \end{aligned} \tag{7}$$

Here,

$$\begin{aligned} U^* &:= \ \text{USC envelope of } U := \text{smallest USC function} \geq U; \\ U_* &:= \ \text{LSC envelope of } U := \text{largest LSC function} \leq U. \end{aligned}$$

Note: g is nonnegative $\Rightarrow \mathcal{U}_g^- \neq \emptyset \quad (u(t,x) \equiv 0$ belongs to $\mathcal{U}_g^-)$.
$\Rightarrow v_g^-(T,x) = g(x)$

Sketch of proof:

1. Assume g is concave. Concavity of g implies $g(X_\cdot^{t,x})$ is a supermartingale. Thus,

$$v_g^-(t,x) \leq U(t,x) = \mathbb{E}^{t,x}[g(X_T^{t,x})] \leq g(x).$$

This implies $U^*(T,x) \leq g(x)$ and $U_*(T,x) \geq v_g^-(T,x) = g(x)$, and thus $U^*(T,x) = U_*(T,x) = g(x)$.

Since a concave function bounded from below is nondecreasing,

$$0 \leq \mathbb{E}^{t,x}[g(X_T^{t,x}) - g(0)] = U(t,x) - g(0) \leq g(x) - g(0).$$

This implies $U^*(t,0) - g(0) \leq 0$ and $U_*(t,0) - g(0) \geq 0$, and thus $U^*(t,0) = U_*(t,0) = g(0)$.

2. $g = g_1 - g_2$, with g_1, g_2 concave. The above result easily extends to this case.

3. The general case: g is continuous. Approximate g by a monotone sequence $\{g^n\}$, with

$$g^n = g_1^n - g_2^n, \quad \text{for some concave } g_1^n, g_2^n.$$

Then, apply results in Step 2 and monotone convergence theorem.

Remark:

- We treat interior smoothness and continuity up to boundary separately.
- **Interior smoothness** requires σ being locally δ-Hölder continuous with $\delta > 0$.
- **continuity up to boundary** requires NO regularity on σ.

Main Results

U is a classical solution

Suppose σ is locally δ-Hölder continuous, with $\delta \in (0, 1]$. Then, for any continuous $g : [0, \infty) \mapsto [0, \infty)$ belonging to $\hat{D}$, the stochastic solution U is a **classical solution** to (4) in $\hat{D}$.

Together with Feynman-Kac formula, we have

U as the unique classical solution

Suppose σ is locally δ-Hölder continuous, with $\delta \in (0, 1]$. Let $g : [0, \infty) \mapsto [0, \infty)$ be continuous and belong to $\hat{D}$. Then, the stochastic solution U is the **unique classical solution** to (4) in $\hat{D}$ if and only if $\int_1^\infty \frac{x}{\sigma^2(x)} dx = \infty$.

Main Results

Comparison theorem

Suppose σ continuous and $\int_1^\infty \frac{x}{\sigma^2(x)}dx = \infty$. Let $u \in \hat{D}$ be a subsolution (resp. $v \in \hat{D}$ be a supersolution) to

$$-\partial_t w - \frac{1}{2}\sigma^2(x)\partial_{xx} w = 0 \quad \text{on } [0,T) \times (0,\infty).$$

If $u \leq v$ on $t = T$ and $x = 0$, then $u \leq v$ on $[0,T] \times [0,\infty)$

- No Hölder continuity of σ is needed.
- To prove a comparison theorem, linear growth on σ is a standard assumption; see e.g. Pham (2009).

  Here, we assume only $\int_1^\infty \frac{x}{\sigma^2(x)}dx = \infty$!

THEOREM

Suppose σ is locally δ-Hölder continuous, with $\delta \in (0,1]$. Then the following are equivalent:

(I) $\int_1^\infty \frac{x}{\sigma^2(x)} dx = \infty$.

(II) X is a true martingale.

(III) (4) admits a unique classical solution in $\hat{D}$ (which is U).

(IV) A comparison theorem for (4) holds among sub(super-)solutions in $\hat{D}$.

- This in particular gives the nontrivial relation "(iii)$\Rightarrow$(iv)".

Suppose σ is continuous only, **without any Hölder continuity**.

$\Rightarrow$ Previous results about smoothness of U no longer holds!

Idea: Approximate σ by $\{\sigma_n\}$ of Hölder continuous functions.

1. $\sigma_n \uparrow \sigma$.
2. σ_n is **locally Hölder continuous, with exponent** $\delta_n \in (0,1]$.
3. for any compact $K \subset (0,\infty)$,

$$\max_{x\in K}\left\{\frac{1}{\sigma_n^2(x)} - \frac{1}{\sigma^2(x)}\right\} < \frac{1}{n}.$$

Lemma

For any (t,x), $X_T^{(n),t,x} \to X_T^{t,x}$ in distribution.

It follows that

$$\mathbb{E}^{t,x}[f(X_T^{(n),t,x})] \to \mathbb{E}^{t,x}[f(X_T^{t,x})] \text{ for any bounded continuous } f.$$

In particular,

$$U_N^M := \mathbb{E}^{t,x}[g(X_T^{(n),t,x}) \wedge M] \to \mathbb{E}^{t,x}[g(X_T^{t,x}) \wedge M] \text{ for any } M > 0.$$

Theorem

Let σ be continuous. For any continuous $g : [0,\infty) \mapsto [0,\infty)$,

$$\lim_{M\to\infty} \lim_{n\to\infty} U_n^M(t,x) = \lim_{M\to\infty} \mathbb{E}^{t,x}[g(X_T^{t,x}) \wedge M] = \mathbb{E}[g(X_T^{t,x})]$$
$$= U(t,x).$$

Now, suppose σ is continuous and $\int_1^\infty \frac{x}{\sigma^2(x)}dx = \infty$.

Since $\sigma_n \uparrow \sigma$, must have $\int_1^\infty \frac{x}{\sigma_n^2(x)}dx = \infty$. Recall that σ_n is locally δ-Hölder continuous with $\delta_n > 0$, we conclude

$U_n^M = \mathbb{E}^{t,x}[g(X_T^{t,x}) \wedge M]$ is the unique classical solution to

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma_n^2(x)\partial_{xx}u = 0, & (t,x) \in [0,T) \times (0,\infty); \\ u(T,x) = g(x) \wedge M, & x \in (0,\infty); \\ u(t,0) = g(0) \wedge M, & t \in [0,T]. \end{cases} \tag{8}$$

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Thank you very much for your attention!
Q & A