**APPM 2460**
**CHAOTIC DYNAMICS**

## 1. Introduction

Today we'll continue our exploration of dynamical systems, focusing in particular upon systems who exhibit a type of strange behavior known as "chaos." We will first look at simple pendulum systems, then we will work our way up to some of the classical chaotic systems.

## 2. Attractors and Limit Cycles

Consider a mass-spring that is unforced and undamped, with mass and spring constant $m = k = 1$. We start the mass at $x(0) = 1$ and $x'(0) = 0$. The IVP for this system is

$$x'' + x = 0, \quad \text{with } x(0) = 1, \ x'(0) = 0.$$

As we did last time, we could rewrite the solution to this system as a vector, $y = (x, v)$. The solution would then be

$$y(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

If we plot this vector in *phase space*, then we see a circle traced out by our system. This is just the parametric path $(\cos(t), -\sin(t))$.
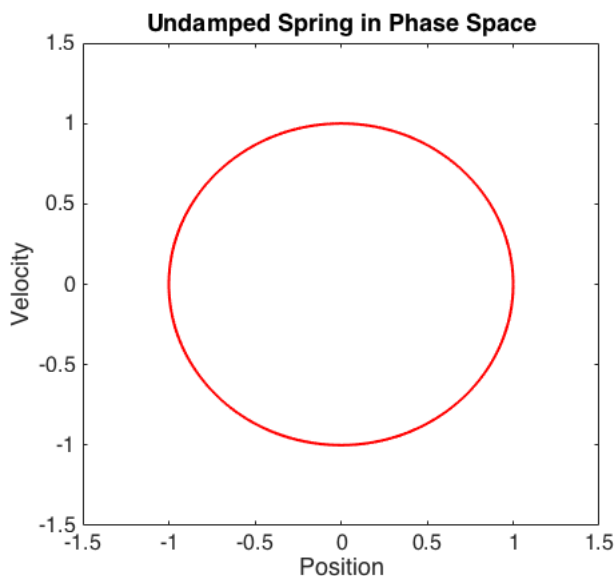


FIGURE 1. An undamped oscillator in phase space. Note that it traces out a circular cycle. It endlessly loops around this circle.

This circle is an example of a limit cycle - it is a cycle that the system repeats endlessly. Let's take a look at a more complicated system. Suppose that we have a system that is *both* damped and driven. Such a system would be

$$x'' + x' + x = \cos(t), \quad \text{with } x(0) = 1.5, \ x'(0) = 0.$$

Loosely speaking, this system *really wants* to be oscillating like $x(t) = \sin(t)$ (note that this solves the differential equation, but doesn't quite satisfy the initial conditions). Note that this would mean

$x'(t) = \cos(t)$, and so we would be back on our circular limit cycle. However, when we start it at $x(0) = 1.5$, it takes some time to settle down to it's comfortable oscillation.

Can we show this using Matlab? First, let's input the system into a function like we learned last time:

```
function yprime = damped_spring(t,y)

    yprime = zeros(2,1);
    yprime(1) = y(2);
    yprime(2) = - y(2) - y(1) + cos(t);

end
```

Then, we can run a script that uses `ode45` to solve the IVP:

```
tspan = [0 20];
y0 = [1.5 0];
[t,y] = ode45(@damped_spring,tspan,y0);
plot(y(:,1),y(:,2))
```

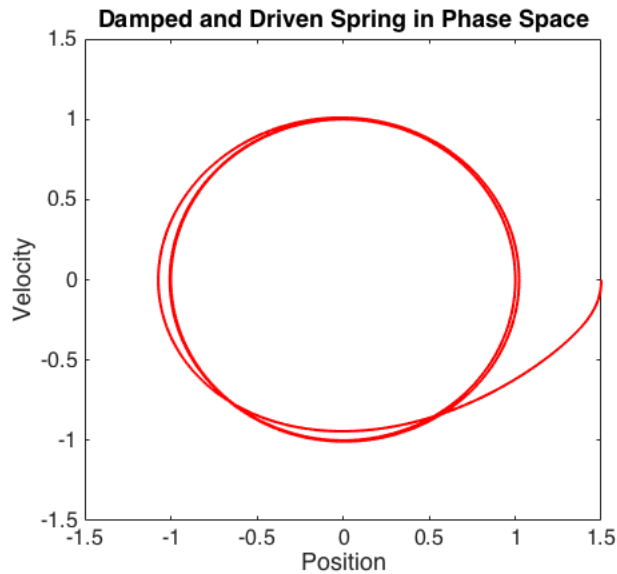When we run this script, we see the following result.



FIGURE 2. A damped and driven oscillator in phase space. The oscillator longs to oscillate at its natural frequency and amplitude of 1, but it must take some time to return to that state, as it begins at $(x, v) = (1.5, 0)$.

Here we clearly see the limit cycle behavior: the oscillator starts well off the curve $(x, v) = (\sin(t), \cos(t))$, but it eventually settles into the groove of the circular limit cycle.

This limit cycle is a nice circle. The fact that it's such a simple curve indicates that the system settles into predictable, repeating behavior after some amount of time. What about a more complicated system?

## 3. CHAOTIC OSCILLATORS

On last week's homework, we saw that if we use the van der Pol system

$$x(t)'' - \mu(1 - x^2(t))x'(t) + x(t) - A\sin(\omega t) = 0$$

with parameters $\mu = 8.53$, $A = 1.2$, and $\omega = 2\pi/10$ and the initial conditions $x(0) = 2$ and $x'(0) = 0$, we get very strange behavior. In particular, we get a pattern that does not quite repeat itself. If we take a look at position versus time as in Figure 3, we can see this quite clearly.
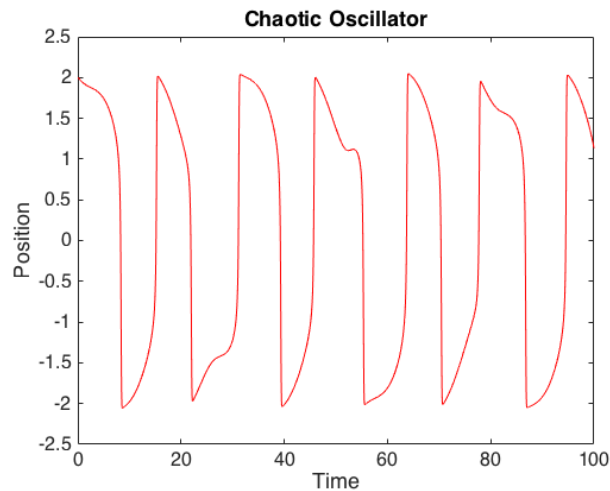
FIGURE 3. A chaotic van der Pol oscillator. Note how, unlike a regular oscillator, it almost **but never quite** repeats itself.

Figure 4 shows this oscillator in phase space, and we see that it almost settles on a nice curve, but not quite. In Figure 5 we zoom in on that weird region on the top of the curve. We observe something that
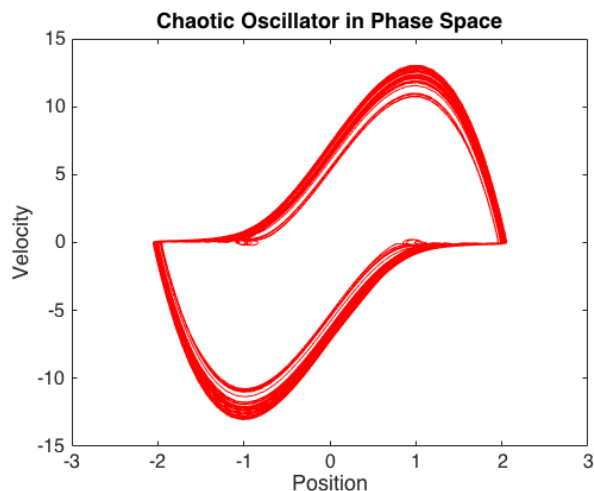


FIGURE 4. A chaotic van der Pol oscillator in phase space.

is very strange. The system is not settling down onto a simple curve like a circle. In fact, we are always *almost* repeating ourselves, but never quite. This is called a **strange attractor**, and is the hallmark of chaotic dynamics.

## 4. CHAOTIC OSCILLATORS IN 3-D

Now let's look at something in three dimensions. We'll examine the famous Lorenz attractor. Consider the following system:

$$x' = \sigma(y - x)$$
$$y' = x(\rho - z) - y$$
$$z' = xy - \beta z$$

with parameters $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. This system is often called the Lorenz system, after Edward Lorenz who studied it extensively. We could write a function that describes this system like so:
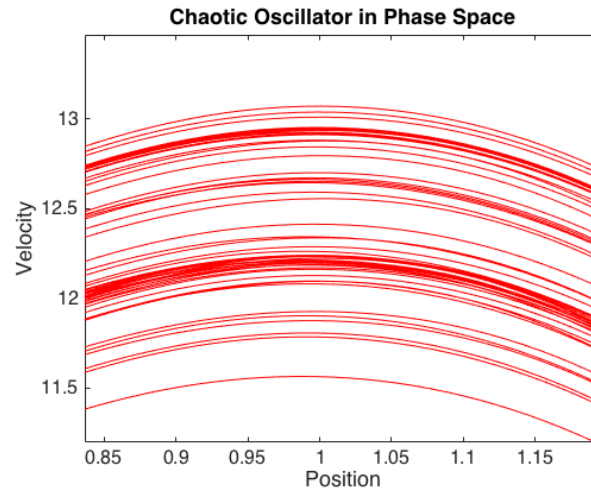
FIGURE 5. Zooming in on the top of the phase-space curve of the van der Pol oscillator.

```
function yprime = lorenz(t,y)

    sigma = 10;
    rho = 28;
    beta = 8/3;

    yprime = zeros(3,1);
    yprime(1) = sigma*(y(2)-y(1));
    yprime(2) = y(1)*(rho-y(3)) - y(2);
    yprime(3) = y(1)*y(2) - beta*y(3);

end
```

and then we could call it in a script like so:

```
tspan = [0 100];
y0 = [1 1 1];
[t,y] = ode45(@lorenz,tspan,y0);
plot3(y(:,1),y(:,2),y(:,3))
```

Note that we have three variables, so we use `plot3` to plot in 3 dimensions. The resulting plot is shown in Figure 6.

## 5. HOMEWORK

For your homework this week, you will simulate and plot the Rossler system. The system is

$$x' = -y - z$$
$$y' = x + ay$$
$$z' = b + z(x - c)$$

Use the parameters $a = b = 0.1$, and $c = 14$. Use initial conditions $x(0) = 15$, $y(0) = 1$, and $z(0) = 0.1$. Use `plot3` to plot the resulting dynamics.
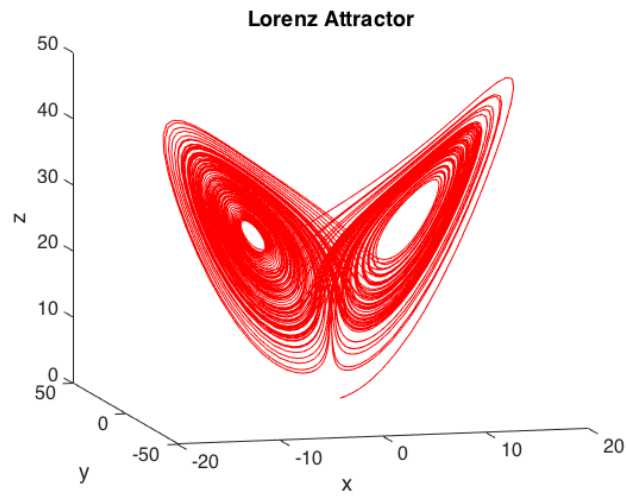
FIGURE 6. The famous Lorenz attractor.