Outperforming The Market Portfolio With A Given Probability

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Outline

- Consider a financial market with a bond $B(\cdot) = 1$ and d stocks $X = (X_1, \cdots, X_d)$ which satisfy for $i = 1; \cdots d$,

$$dX_i(t) \quad = X_i(t) \left(b_i(X(t))dt + \sum_{k=1}^{d} s_{ik}(X(t))dW_k(t) \right). \tag{1}$$

- Let $\mathcal{H}$ denote the collection of all trading strategies.
- For each $\pi \in \mathcal{H}$ and initial wealth $y \geq 0$, the associated wealth process will be denoted by $Y^{y,\pi}(\cdot)$.

The Problem

- In this paper, we want to determine and characterize

The Problem

$$V(T,x,p) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t.} \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\}$$

, where $g : (0,\infty)^d \mapsto \mathbb{R}_+$ is a measurable function.

Related Work

- In the case where $p = 1$ and $g(x) = x_1 + \cdots + x_d$,

  $$V(T, x, 1) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t. } Y^{y,\pi}(T) \geq g(X(T)) \text{ a.s.}\}.$$

  In Fernholz and Karatzas (2010), a PDE characterization for $\tilde{V}(T, x, 1) := V(T, x, 1)/g(x)$ was derived when $V(T, x, 1)$ is assumed to be smooth.

- In Bouchard, Elie and Touzi (2009), a PDE characterization of $V(t, x, p)$ was derived.
  - Assumptions: rather strong, e.g. existence of a unique strong solution of (1);
  - main tool used: Geometric dynamic programming principle.

  Under the No-Arbitrage condition, they recovered the solution of quantile hedging problem proposed in Follmer and Leukert (1999).

Related Work

- In our paper, we will also have a PDE characterization for $V(t,x,p)$, but
  - We only assume the existence of a weak solution of (1) that is unique in distribution;
  - We admit arbitrage in our model.

Assumptions

Assumption 2.1

- Let $b_i : (0, \infty)^d \to \mathbb{R}$ and $s_{ik} : (0, \infty)^d \to \mathbb{R}$ be continuous functions and $b(\cdot) = (b_1(\cdot), \cdots, b_d(\cdot))'$ and $s(\cdot) = (s_{ij}(\cdot))_{1 \le i,j \le d}$, which we assume to be invertible for all $x \in (0, \infty)^d$.
- We also assume that (1) has a weak solution that is unique in distribution for every initial value.
- Let $\theta(\cdot) := s^{-1}(\cdot)b(\cdot)$, $a_{ij}(\cdot) := \sum_{i=1}^{d} s_{ik}(\cdot)s_{jk}(\cdot)$ s atisfy

$$\sum_{i}^{d} \int_0^T \left(|b_i(X(t))| + a_{ii}(X(t)) + \theta_i^2(X(t))\right) < \infty. \qquad (2)$$

Consequences of Assumptions

- We denote by $\mathbb{F}$ the augmentation of the natural filtration of $X(\cdot)$.
- Thanks to Assumption 2.1,
  - every local martingale of $\mathbb{F}$ has the martingale representation property with respect to $W(\cdot)$ (adapted to $\mathbb{F}$).
  - the solution of (1) takes values in the positive orthant
  - the exponential local martingale

$$Z(t) := \exp\left\{-\int_0^t \theta(X(s))' dW(s) - \frac{1}{2}\int_0^t |\theta(X(s))|^2 ds\right\}, \tag{3}$$

the so-called *deflator* is well defined. We do not exclude the possibility that $Z(\cdot)$ is a strict local martingale.

Outline

- Let $g : (0,\infty)^d \to \mathbb{R}_+$ be a measurable function satisfying

$$\mathbb{E}[Z(T)g(X(T))] < \infty. \tag{4}$$

- We want to determine

$$V(T,x,p) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\}, \tag{5}$$

for $p \in [0,1]$.

- We will always assume Assumption 2.1 and (4) hold.

Lemma 3.1

We will present a probabilistic characterization of $V(T, x, p)$.

Lemma 3.1

Given $A \in \mathcal{F}_T$,

(I) if $\mathbb{P}(A) \geq p$, then

$$V(T, x, p) \leq \mathbb{E}[Z(T)g(X(T))1_A].$$

(II) if $\mathbb{P}(A) = p$ and

$$\operatorname{ess\,sup}_A \{Z(T)g(X(T))\} \leq \operatorname{ess\,inf}_{A^c} \{Z(T)g(X(T))\}, \quad (6)$$

then

$$V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A]. \quad (7)$$

Propositions 3.1 & 3.3

Proposition 3.1

Fix $(x, p) \in (0, \infty)^d \times [0, 1]$. There exists $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (6). As a result, we have

$$V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A].$$

Let

$$\mathcal{M} := \{\varphi : \Omega \to [0, 1] \text{ is } \mathcal{F}_T \text{ measurable s.t. } \mathbb{E}[\varphi] \geq p\}.$$

Using Proposition 3.1, we give an alternative representation of V

Proposition 3.3

$V(T, x, p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi]$.

This will facilitate the PDE characterization in the next section.

Outline

The Value Function U

- $V(T,x,p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi]$.
- Define the value function

$$U(t,x,p) := \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))\varphi], \qquad (8)$$

where $X^{t,x}(\cdot)$ denotes the solution of (1) starting from x at time t, and $Z^{t,x,z}(\cdot)$ denotes the solution of

$$dZ(s) = -Z(s)\theta(X^{t,x}(s))' dW(s), \ Z(t) = z. \qquad (9)$$

- $V(T,x,p) = U(0,x,p)$.

The Plan...

$$U(t, x, p) \longrightarrow w(t, x, q)$$

1. : Legendre transform of U w.r.t. p.

The Functions w and $\widetilde{w}$

- Consider the Legendre tranform of U with respect to the p variable

$$w(t,x,q) := \sup_{p\in[0,1]} \{pq - U(t,x,p)\}, \tag{10}$$

- Define the process $Q^{t,x,q}(\cdot)$ by

$$Q^{t,x,q}(\cdot) := \frac{1}{Z^{t,x,(1/q)}(\cdot)}, \; q \in (0,\infty). \tag{11}$$

Then we see from (9) that $Q(\cdot)$ satisfies

$$\frac{dQ(s)}{Q(s)} = |\theta(X^{t,x}(s))|^2 ds + \theta(X^{t,x}(s))' dW(s), \; Q^{t,x,q}(t) = q. \tag{12}$$

The Functions w and $\widetilde{w}$ (conti.)

- Define the function

$$\widetilde{w}(t,x,q) \quad := \quad \mathbb{E}[Z^{t,x,1}(T)(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+]$$

- By Proposition 3.1, we can show that $w = \widetilde{w}$.
- Interpret $\widetilde{w}$ as the superhedging price of $(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+$; then it potentially solves

$$\partial_t \widetilde{w} + \frac{1}{2}\mathrm{Tr}(\sigma\sigma' D_x^2 \widetilde{w}) + \frac{1}{2}|\theta|^2 q^2 D_q^2 \widetilde{w} + q\mathrm{Tr}(\sigma\theta D_{xq}\widetilde{w}) = 0. \quad (13)$$

where $\sigma_{ik}(x) := s_{ik}(x)x_i$.

THE FUNCTIONS w AND $\widetilde{w}$ (CONTI.)

- However, the covariance matrix is degenerate! Indeed, setting

$$v(\cdot) := \left[\frac{s(\cdot)_{d\times d}}{\theta(\cdot)'_{1\times d}} \right],$$

degeneracy can be seen by observing that $v(x)v(x)'$ is only positive semi-definite for all $x \in (0,\infty)^d$.

THE PLAN...

$$
\begin{array}{ccc}
U(t,x,p) & \longrightarrow & w(t,x,q) = \widetilde{w}(t,x,q) \\
 & & \big\downarrow \\
 & & \widetilde{w}_{\varepsilon}(t,x,q)
\end{array}
$$

1. : Legendre transform of U w.r.t. p.
2. : Elliptic regularization for $\widetilde{w}$.

Elliptic Regularization

- For any $\varepsilon > 0$, introduce the process $Q_\varepsilon^{t,x,q}(\cdot)$ which satisfies

$$\frac{dQ_\varepsilon(s)}{Q_\varepsilon(s)} = |\theta(X^{t,x}(s))|^2 ds + \theta(X^{t,x}(s))' dW(s) + \varepsilon dB(s), \quad (14)$$

  where $B(\cdot)$ is a one-dimensional B.M. independent of $W(\cdot)$.
- Define the function

$$\widetilde{w}_\varepsilon(t,x,q) := \bar{\mathbb{E}}[Z^{t,x,1}(T)(Q_\varepsilon^{t,x,q}(T) - g(X^{t,x}(T)))^+],$$

Assumption 4.1

θ_i and σ_{ij} are, for all $i, j \in \{1, \cdots, d\}$, locally Lipschitz.

Applying Ruf [2010, Theorem 2], we obtain

Lemma 4.1

Under Assumption 4.1, $\widetilde{w}_\varepsilon \in \mathcal{C}^{1,2,2}((0,T) \times (0,\infty)^d \times (0,\infty))$ satisfies the PDE

$$\partial_t \widetilde{w}_\varepsilon + \frac{1}{2}\mathrm{Tr}(\sigma\sigma' D_x^2 \widetilde{w}_\varepsilon) + \frac{1}{2}(|\theta|^2 + \varepsilon^2) q^2 D_q^2 \widetilde{w}_\varepsilon + q\mathrm{Tr}(\sigma\theta D_{xq}\widetilde{w}_\varepsilon) = 0, \tag{15}$$

with the boundary condition

$$\widetilde{w}_\varepsilon(T, x, q) = (q - g(x))^+. \tag{16}$$

The Plan...

$$\begin{array}{ccc} U(t,x,p) & \longrightarrow & w(t,x,q) = \widetilde{w}(t,x,q) \\ & & \downarrow \\ U_\varepsilon(t,x,p) & \longleftarrow & \widetilde{w}_\varepsilon(t,x,q) \end{array}$$

1. : Legendre transform of U w.r.t. p.
2. : Elliptic regularization for $\widetilde{w}$.
3. : Legendre transform of $\widetilde{w}_\varepsilon$ w.r.t. q.

The PDE for U_ε

- Consider the Legendre transform of $\widetilde{w}_\varepsilon$ w.r.t. the q variable

$$U_\varepsilon(t,x,p) := \sup_{q\in\mathbb{R}}\{pq - \widetilde{w}_\varepsilon(t,x,q)\} = \sup_{q\geq 0}\{pq - \widetilde{w}_\varepsilon(t,x,q)\}.$$

- Introduce a geometric Brownian motion $L_\varepsilon(\cdot)$ which satisfies

$$dL_\varepsilon(s) = \varepsilon L_\varepsilon(s)dB(s),\ \ s\in[t,T] \text{ and } L(t)=1.$$

Then $L_\varepsilon(\cdot)$ attains any interval on the positive real line with positive probability. Using this property, we show that $\widetilde{w}_\varepsilon(t,x,q)$ is strictly convex in q.

The PDE for U_ε (conti.)

Proposition 4.4

Under Assumption 4.1, $U_\varepsilon \in \mathcal{C}^{1,2,2}((0,T)\times(0,\infty)^d\times(0,1))$ satisfies

$$\begin{aligned} 0 = {} & \partial_t U_\varepsilon + \frac{1}{2} Tr[\sigma\sigma' D_{xx} U_\varepsilon] \\ & + \inf_{a\in\mathbb{R}^d}\left((D_{xp}U_\varepsilon)'\sigma a + \frac{1}{2}|a|^2 D_{pp}U_\varepsilon - \theta' a D_p U_\varepsilon\right) \\ & + \inf_{b\in\mathbb{R}^d}\left(\frac{1}{2}|b|^2 D_{pp}U_\varepsilon - \varepsilon D_p U_\varepsilon \mathbf{1}' b\right), \end{aligned} \tag{17}$$

where $\mathbf{1} := (1,\cdots,1)' \in \mathbb{R}^d$, with the boundary condition

$$U_\varepsilon(T,x,p) = pg(x). \tag{18}$$

THE PLAN...

$$
\begin{array}{ccc}
U(t,x,p) & \longrightarrow & w(t,x,q) = \widetilde{w}(t,x,q) \\
\uparrow & & \downarrow \\
U_\varepsilon(t,x,p) & \longleftarrow & \widetilde{w}_\varepsilon(t,x,q)
\end{array}
$$

1. : Legendre transform of U w.r.t. p.
2. : Elliptic regularization for $\widetilde{w}$.
3. : Legendre transform of $\widetilde{w}_\varepsilon$ w.r.t. q.
4. : "$\liminf_{\varepsilon \to 0} U_\varepsilon = U$" & "Stability of viscosity solutions."

The PDE for U

- For any $(x, \beta, \gamma, \lambda) \in (0, \infty)^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, define

$$G(x, \beta, \gamma, \lambda) := \inf_{a \in \mathbb{R}^d} \left(\lambda' \sigma(x) a + \frac{1}{2} |a|^2 \gamma - \beta \theta(x)' a \right).$$

Also, consider the lower semicontinuous envelope of G

$$G_*(x, \beta, \gamma, \lambda) := \liminf_{(\tilde{x}, \tilde{\beta}, \tilde{\gamma}, \tilde{\lambda}) \to (x, \beta, \gamma, \lambda)} G(\tilde{x}, \tilde{\beta}, \tilde{\gamma}, \tilde{\lambda}).$$

The PDE for U (conti.)

By using the stability of viscosity solutions, we have

Proposition 4.5

Under Assumption 4.1, U is a lower semicontinuous viscosity supersolution of

$$0 \geq \partial_t U + \frac{1}{2} Tr[\sigma\sigma' D_{xx} U] + G_*(x, D_p U, D_{pp} U, D_{xp} U), \tag{19}$$

for $(t, x, p) \in (0, T) \times (0, \infty)^d \times (0, 1)$, with the boundary condition

$$U(T, x, p) = pg(x), \tag{20}$$

Uniqueness??

Characterize U further

We characterize U as the smallest nonnegative l.s.c. viscosity supersolution to (19) with the boundary condition (20) among a particular set of functions.

Proposition 4.7

Suppose Assumption 4.1 holds. Let
$u : [0, T] \times (0, \infty)^d \times [0, 1] \mapsto [0, \infty)$ be such that

- $u(t, x, 0) = 0$,
- $u(t, x, p)$ is convex in p,
- the Legendre transform of u w.r.t. p is continuous on $[0, T] \times (0, \infty)^d \times (0, \infty)$.

Then, if u is a lower semicontinuous viscosity supersolution to (19) on $(0, T) \times (0, \infty)^d \times (0, 1)$ with the boundary condition (20), then $u \geq U$.

References

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Thank you very much for your attention!
Q & A