On the Multi-Dimensional Controller and Stopper Games

Yu-Jui Huang

Joint work with Erhan Bayraktar
University of Michigan, Ann Arbor

SIAM Conference on Financial Mathematics and Engineering
Minneapolis
July 10, 2012

We consider the robust (worst-case) optimal stopping problem:

$$V(t,x) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\Big[\int_t^\tau e^{-\int_t^s c(u, X_u^{t,x,\alpha}) du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^\tau c(u, X_u^{t,x,\alpha}) du} g(X_\tau^{t,x,\alpha}) \Big],$$

where $\mathcal{A}_t$: set of controls, $\mathcal{T}^t_{t,T}$: set of stopping times.

- $f(s, X_s^\alpha, \alpha_s)$: running cost at time s.
- $g(X_\tau^\alpha)$: terminal cost at time τ.
- $c(s, X_s^\alpha)$: discount rate at time s.
- X^α: a controlled state process.

Think of this as a controller-stopper game between

us (stopper) and nature (controller)!

If "Stopper" acts first: Instead of choosing one single stopping time, he would like to employ a strategy $\pi : \mathcal{A}_t \mapsto \mathcal{T}_{t,T}^t$.

$$U(t,x) := \inf_{\pi \in \Pi_{t,T}^t} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[\int_t^{\pi[\alpha]} e^{-\int_t^s c(u, X_u^{t,x,\alpha})du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^{\pi[\alpha]} c(u, X_u^{t,x,\alpha})du} g(X_{\pi[\alpha]}^{t,x,\alpha}) \right],$$

where Π is the set of strategies $\pi : \mathcal{A}_t \mapsto \mathcal{T}_{t,T}^t$.

If "Controller" acts first: nature does NOT use strategies.

$$V(t,x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}[\cdots].$$

By definition, $V \leq U$. We say **the game has a value** if $U = V$.

Related Work

The controller-stopper game is closely related to some common problems in mathematical finance:

- pricing American contingent claims, see e.g. Karatzas & Kou [1998], Karatzas & Wang [2000] and Karatzas & Zamfirescu [2005].
- minimizing the probability of lifetime ruin, see Bayraktar & Young [2011].

But, the game itself has been studied to a great extent only in some special cases.

Related Work

One-dimensional case: Karatzas and Sudderth [2001] study the case where X^α moves along an interval on $\mathbb{R}$.

- they show that the game has a value;
- they construct a saddle-point of optimal strategies (α^*, τ^*).

Difficult to extend their results to higher dimensions (their techniques rely heavily on optimal stopping theorems for one-dimensional diffusions).

Multi-dimensional case: Karatzas and Zamfirescu [2008] develop a martingale approach to deal with this. But, require some **strong** assumptions:

- the diffusion term of X^α has to be non-degenerate, and it cannot be controlled!

Our Goal

We intend to investigate a much more general multi-dimensional controller-stopper game in which

- both the drift and the diffusion terms of X^α can be controlled;
- the diffusion term can be degenerate.

Main Result: Under appropriate conditions,

- the game has a value (i.e. $U = V$);
- the value function is the unique viscosity solution to an obstacle problem of an HJB equation.

$$U^* \geq U \geq V \geq V_*$$

1. Weak DPP for U

$\Rightarrow$ **subsolution** property of $\mathbf{U^*}$

2. Weak DPP for V

$\Rightarrow$ **supersolution** property of $\mathbf{V_*}$

3. A comparison result $\Rightarrow V_* \geq U^*$ (supersol. $\geq$ subsol.)
$\Rightarrow U^* = V_* \Rightarrow U = V$, i.e. **the game has a value.**

The Set-up

Consider a fixed time horizon $T > 0$.

- $\Omega := C([0, T]; \mathbb{R}^d)$.
- $W = \{W_t\}_{t \in [0,T]}$: the canonical process, i.e. $W_t(\omega) = \omega_t$.
- $\mathbb{P}$: the Wiener measure defined on Ω.
- $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$: the $\mathbb{P}$-augmentation of $\sigma(W_s, s \in [0, T])$.

For each $t \in [0, T]$, consider

- $\mathbb{F}^t$: the $\mathbb{P}$-augmentation of $\sigma(W_{t \vee s} - W_t, s \in [0, T])$.
- $\mathcal{T}^t$:=$\{\mathbb{F}^t$-stopping times valued in $[0, T]$ $\mathbb{P}$-a.s.$\}$.
- $\mathcal{A}_t$:=$\{\mathbb{F}^t$-progressively measurable M-valued processes$\}$, where M is a separable metric space.
- Given $\mathbb{F}$-stopping times τ_1, τ_2 with $\tau_1 \leq \tau_2$ $\mathbb{P}$-a.s., define $\mathcal{T}^t_{\tau_1,\tau_2}$:=$\{\tau \in \mathcal{T}^t$ valued in $[\tau_1, \tau_2]$ $\mathbb{P}$-a.s.$\}$.

Assumptions on b and σ

Given $\tau \in \mathcal{T}$, $\xi \in \mathcal{L}_d^p$ which is $\mathcal{F}_\tau$-measurable, and $\alpha \in \mathcal{A}$, let $X^{\tau,\xi,\alpha}$ denote a $\mathbb{R}^d$-valued process satisfying the SDE:

$$dX_t^{\tau,\xi,\alpha} = b(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dt + \sigma(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dW_t, \tag{1}$$

with the initial condition $X_\tau^{\tau,\xi,\alpha} = \xi$ a.s.

Assume: $b(t,x,u)$ and $\sigma(t,x,u)$ are deterministic Borel functions, and continuous in (x,u); moreover, $\exists\ K > 0$ s.t. for $t \in [0,T]$, $x, y \in \mathbb{R}^d$, and $u \in M$

$$\begin{aligned} &|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \leq K|x-y|, \\ &|b(t,x,u)| + |\sigma(t,x,u)| \leq K(1+|x|), \end{aligned} \tag{2}$$

This implies for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, (1) admits a unique strong solution $X_\cdot^{t,x,\alpha}$.

Assumptions on f, g, and c

f and g are rewards, c is the discount rate $\Rightarrow$ assume $f, g, c \geq 0$.

In addition, Assume:

- $f : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}$ is Borel measurable, and $f(t, x, u)$ continuous in (x, u), and continuous in x uniformly in $u \in M$.
- $g : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous,
- $c : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ is continuous and bounded above by some real number $\bar{c} > 0$.
- f and g satisfy a polynomial growth condition

$$|f(t, x, u)| + |g(x)| \leq K(1 + |x|^{\bar{p}}) \text{ for some } \bar{p} \geq 1. \tag{3}$$

Reduction to the Mayer form

Set $F(x, y, z) := z + yg(x)$. Observe that

$$V(t,x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\left[Z^{t,x,1,0,\alpha}_\tau + Y^{t,x,1,\alpha}_\tau g(X^{t,x,\alpha}_\tau)\right]$$
$$= \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\left[F(\mathbf{X}^{t,x,1,0,\alpha}_\tau)\right],$$

where $\mathbf{X}^{t,x,y,z,\alpha}_\tau := (X^{t,x,\alpha}_\tau, Y^{t,x,y,\alpha}_\tau, Z^{t,x,y,z,\alpha}_\tau)$. Similarly,

$$U(t,x) = \inf_{\pi \in \Pi^t_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[F(\mathbf{X}^{t,x,1,0,\alpha}_{\pi[\alpha]})\right].$$

More generally, for any $(x, y, z) \in \mathcal{S} := \mathbb{R}^d \times \mathbb{R}^2_+$, define

$$\bar{V}(t,x,y,z) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\left[F(\mathbf{X}^{t,x,y,z,\alpha}_\tau)\right].$$
$$\bar{U}(t,x,y,z) := \inf_{\pi \in \Pi^t_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\left[F(\mathbf{X}^{t,x,y,z,\alpha}_{\pi[\alpha]})\right].$$

For $(t, x, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^d$, define

$$H^a(t, x, p, A) := -b(t, x, a) - \frac{1}{2} Tr[\sigma\sigma'(t, x, a)A] - f(t, x, a),$$

and set

$$H(t, x, p, A) := \inf_{a \in M} H^a(t, x, p, A).$$

Proposition

The function U^* is a viscosity subsolution on $[0, T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max\left\{c(t, x)w - \frac{\partial w}{\partial t} + H_*(t, x, D_x w, D_x^2 w), w - g(x)\right\} \leq 0.$$

Sketch of proof:

1. **Assume the contrary:** $\exists$ smooth h, $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ s.t.

$$0 = (U^* - h)(t_0, x_0) > (U^* - h)(t, x),\ \forall\ (t, x) \in [0, T) \times \mathbb{R}^d \setminus (t_0, x_0);$$

$$\max\left\{ c(t_0, x_0)h - \frac{\partial h}{\partial t} + H_*(t_0, x_0, D_x h, D_x^2 h), h - g(x_0) \right\}(t_0, x_0) > 0.$$

2. **Applying Itô's rule locally at $(\mathbf{t_0}, \mathbf{x_0})$**, we eventually get

$$U(\hat{t}, \hat{x}) > \mathbb{E}\left[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha} h(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s) ds \right] + \frac{\eta}{2},$$

for any $\alpha \in \mathcal{A}_{\hat{t}}$, where

$$\theta^\alpha := \inf\left\{ s \geq \hat{t} \,\middle|\, (s, X_s^{\hat{t},\hat{x},\alpha}) \notin B_r(t_0, x_0) \right\} \in \mathcal{T}_{\hat{t},T}^{\hat{t}}.$$

HOW TO GET A CONTRADICTION TO THIS?

By the definition of U,

$$
\begin{aligned}
U(\hat{t},\hat{x}) &\leq \sup_{\alpha\in\mathcal{A}_{\hat{t}}} \mathbb{E}\left[F\left(\mathbf{X}^{\hat{t},\hat{x},1,0,\alpha}_{\pi^*[\alpha]}\right)\right]\\
&\leq \mathbb{E}\left[F\left(\mathbf{X}^{\hat{t},\hat{x},1,0,\hat{\alpha}}_{\pi^*[\hat{\alpha}]}\right)\right] + \frac{\eta}{4}, \text{ for some } \hat{\alpha}\in\mathcal{A}_{\hat{t}}.\\
&\leq \mathbb{E}\left[Y^{\hat{t},\hat{x},1,\hat{\alpha}}_{\theta^{\hat{\alpha}}} h(\theta, X^{\hat{t},\hat{x},\hat{\alpha}}_{\theta^{\hat{\alpha}}}) + Z^{\hat{t},\hat{x},1,0,\hat{\alpha}}_{\theta^{\hat{\alpha}}}\right] + \frac{\eta}{4} + \frac{\eta}{4}.
\end{aligned}
$$

The BLUE PART is the WEAK DPP we want to prove!

Weak DPP for U

Proposition (Weak DPP for U)

Fix $(t,\mathbf{x}) \in [0,T] \times \mathcal{S}$ and $\varepsilon > 0$. For any $\pi \in \Pi^t_{t,T}$ and $\varphi \in LSC([0,T] \times \mathbb{R}^d)$ with $\varphi \geq U$, $\exists\ \pi^* \in \Pi^t_{t,T}$ s.t. $\forall \alpha \in \mathcal{A}_t$,

$$\mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^*[\alpha]})\right] \leq \mathbb{E}\left[Y^{t,x,y,\alpha}_{\pi[\alpha]}\varphi\left(\pi[\alpha], X^{t,x,\alpha}_{\pi[\alpha]}\right) + Z^{t,x,y,z,\alpha}_{\pi[\alpha]}\right] + 4\varepsilon.$$

To prove this weak DPP, we need

Lemma

Fix $t \in [0,T]$. For any $\pi \in \Pi^t_{t,T}$, $L^\pi : [0,t] \times \mathcal{S} \mapsto \mathbb{R}$ defined by $L^\pi(s,\mathbf{x}) := \sup_{\alpha \in \mathcal{A}_s} \mathbb{E}\left[F(\mathbf{X}^{s,\mathbf{x},\alpha}_{\pi[\alpha]})\right]$ is continuous.

Idea of Proof: Generalize the arguments in Krylov[1980] for control problems with fixed horizon to our case with random horizon.

Sketch of proof for "Weak DPP for U":

1. **Separate $[0,T]\times\mathcal{S}$ into small pieces.** Since $[0,T]\times\mathcal{S}$ is Lindelöf, take $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ s.t. $\bigcup_{i\in\mathbb{N}} B(t_i,x_i;r^{(t_i,x_i)}) = (0,T]\times\mathcal{S}$,

$$\text{with } B(t_i,x_i;r^{(t_i,x_i)}) := (t_i - r^{(t_i,x_i)}, t_i] \times B_{r^{(t_i,x_i)}}(x_i).$$

Take a disjoint subcovering $\{A_i\}_{i\in\mathbb{N}}$ s.t. $(t_i,x_i)\in A_i$.

2. **Pick ε-optimal strategy $\pi^{(t_i,x_i)}$ in each A_i.** For each (t_i,x_i), by def. of $\bar{U}$, $\exists\ \pi^{(t_i,x_i)} \in \Pi^{t_i}_{t_i,T}$ s.t.

$$\sup_{\alpha\in\mathcal{A}_{t_i}} \mathbb{E}\left[F(\mathbf{X}^{t_i,x_i,\alpha}_{\pi^{(t_i,x_i)}[\alpha]})\right] \le \bar{U}(t_i,x_i)+\varepsilon.$$

Set $\bar{\varphi}(t,x,y,z) := y\varphi(t,x)+z$. For any $(t',x')\in A_i$,

$$\begin{aligned} L^{\pi^{(t_i,x_i)}}(t',x') &\underset{\text{usc}}{\le} L^{\pi^{(t_i,x_i)}}(t_i,x_i)+\varepsilon \le \bar{U}(t_i,x_i)+2\varepsilon \\ &\le \bar{\varphi}(t_i,x_i)+2\varepsilon \underset{\text{lsc}}{\le} \bar{\varphi}(t',x')+3\varepsilon. \end{aligned} \tag{4}$$

3. **Paste $\pi^{(t_i,x_i)}$ together.** For any $n \in \mathbb{N}$, set $B^n := \cup_{1\le i\le n} A_i$ and define $\pi^n \in \Pi^t_{t,T}$ by

$$\pi^n[\alpha] := T 1_{(B^n)^c}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}) + \sum_{i=1}^{n} \pi^{(t_i,x_i)}[\alpha] 1_{A_i}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}).$$

4. **Estimations.**

$$\begin{aligned}
&\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^n[\alpha]})] \\
&\quad = \mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^n[\alpha]}) 1_{B^n}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]})\right] + \mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{T}) 1_{(B^n)^c}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]})\right] \\
&\quad \le \mathbb{E}[\bar{\varphi}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]})] + 3\varepsilon + \varepsilon,
\end{aligned}$$

where RED PART follows from (4) and BLUE PART holds for $n \ge n^*(\alpha)$.

5. **Construct the desired strategy** π^***.** Define $\pi^* \in \Pi_{t,T}^t$ by

$$\pi^*[\alpha] := \pi^{n^*(\alpha)}[\alpha].$$

Then we get

$$\begin{aligned}\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^*[\alpha]})] &\leq \mathbb{E}[\bar{\varphi}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]})] + 4\varepsilon \\ &= \mathbb{E}[Y^{t,x,y,\alpha}_{\pi[\alpha]}\varphi(\theta, X^{t,x,\alpha}_{\pi[\alpha]}) + Z^{t,x,y,z,\alpha}_{\pi[\alpha]}] + 4\varepsilon.\end{aligned}$$

Done with the proof of Weak DPP for U!
Done with the proof of the subsolution property of U^*!

Supersolution Property of V_*

Proposition (Weak DPP for V)

Fix $(t,\mathbf{x}) \in [0,T] \times \mathcal{S}$ and $\varepsilon > 0$. For any $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^t$ and $\varphi \in USC([0,T] \times \mathbb{R}^d)$ with $\varphi \le V$,

(I) $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$;

(II) If, moreover, $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$, then there exists $\alpha^* \in \mathcal{A}_t$ with $\alpha_s^* = \alpha_s$ for $s \in [t,\theta]$ s.t. for any $\tau \in \mathcal{T}_{t,T}^t$,

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^*})] \ge \mathbb{E}[Y_{\tau\wedge\theta}^{t,x,y,\alpha}\varphi(\tau\wedge\theta, X_{\tau\wedge\theta}^{t,x,\alpha}) + Z_{\tau\wedge\theta}^{t,x,y,z,\alpha}] - 4\varepsilon.$$

Proposition

The function V_* is a viscosity supersolution on $[0,T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max\left\{c(t,x)w - \frac{\partial w}{\partial t} + H(t,x,D_x w, D_x^2 w),\ w - g(x)\right\} \ge 0.$$

Comparison

To state an appropriate comparison result, we assume

A. for any $t, s \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$|b(t,x,u)-b(s,y,u)|+|\sigma(t,x,u)-\sigma(s,y,u)| \leq K(|t-s|+|x-y|).$$

B. $f(t,x,u)$ is uniformly continuous in (t,x), uniformly in $u \in M$.

The conditions **A** and **B**, together with the linear growth condition on b and σ, imply that the function H is continuous, and thus $H = H_*$.

Proposition (Comparison)

Assume **A** and **B**. Let u (resp. v) be an USC viscosity subsolution (resp. a LSC viscosity supersolution) with polynomial growth condition to (19), such that $u(T,x) \leq v(T,x)$ for all $x \in \mathbb{R}^d$. Then $u \leq v$ on $[0, T) \times \mathbb{R}^d$.

Lemma

For all $x \in \mathbb{R}^d$, $V_*(T, x) \geq g(x)$.

Theorem

Assume **A** and **B**. Then $U^* = V_*$ on $[0, T] \times \mathbb{R}^d$. In particular, $U = V$ on $[0, T] \times \mathbb{R}^d$, i.e. the game has a value, which is the unique viscosity solution to (19) with terminal condition $w(T, x) = g(x)$ for $x \in \mathbb{R}^d$.

$$U^* \geq U \geq V \geq V_*$$

1. Weak DPP for U

$\Rightarrow$ **subsolution** property of $\mathbf{U^*}$

2. Weak DPP for V

$\Rightarrow$ **supersolution** property of $\mathbf{V_*}$

3. A comparison result $\Rightarrow V_* \geq U^*$ (supersol. $\geq$ subsol.)
$\Rightarrow U^* = V_* \Rightarrow U = V$, i.e. **the game has a value.**

No a priori regularity needed!
(U and V don't even need to be measurable!)
No measurable selection needed!

References I

- E. Bayraktar and V.R. Young, *Proving Regularity of the Minimal Probability of Ruin via a Game of Stopping and Control*, Finance and Stochastics, 15 No.4 (2011), pp. 785–818.
- B. Bouchard, and N. Touzi, *Weak Dynamic Programming Principle for Viscosity Solutions*, SIAM Journal on Control and Optimization, 49 No.3 (2011), pp. 948–962.
- I. Karatzas and S.G. Kou, *Hedging American Contingent Claims with Constrained Portfolios*, Finance & Stochastics, 2 (1998), pp. 215–258.
- I. Karatzas and W.D. Sudderth, *The Controller-and-stopper Game for a Linear Diffusion*, The Annals of Probability, 29 No.3 (2001), pp. 1111–1127.

References II

- I. Karatzas and H. Wang, *A Barrier Option of American Type*, Applied Mathematics and Optimization, 42 (2000), pp. 259–280.
- I. Karatzas and I.-M. Zamfirescu, *Game Approach to the Optimal Stopping Problem*, Stochastics, 8 (2005), pp. 401–435.
- I. Karatzas and I.-M. Zamfirescu, *Martingale Approach to Stochastic Differential Games of Control and Stopping*, The Annals of Probability, 36 No.4 (2008), pp. 1495–1527.
- N.V. Krylov, *Controlled Diffusion Processes*, Springer-Verlag, New York (1980).

Thank you very much for your attention!
Q & A