# 1 Introduction

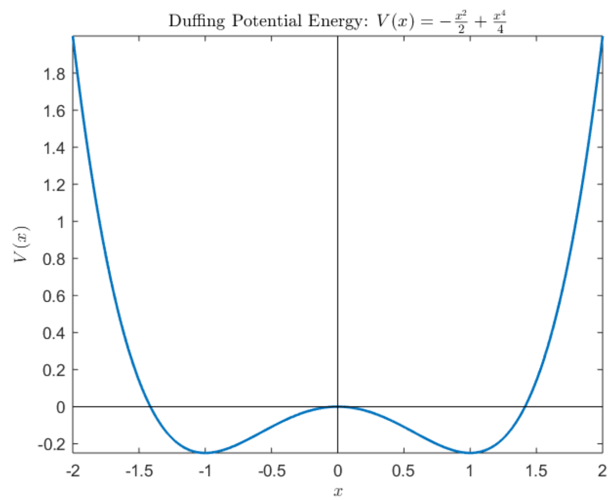
The Duffing Equation is an externally forced and damped oscillator equation that exhibits a range of interesting dynamic behavior in its solutions. While, for many parameter values, the solutions of the system represent a mass-spring system whose response to displacement from equilibrium is characterized by a restoring force exhibiting both linear and cubic features, the system's solutions readily transition to chaotic behavior. This report will explore the Duffing Equation in both the chaotic and non-chaotic regimes.

# 2 The Non-Chaotic Duffing Equation

1. The potential energy function used to derive the Duffing Equation is

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$$

and is visualized as




We can see from the plot that the positions of the minimum values of the potential occur at $x = \pm 1$ and take the values $V(x = \pm 1) = -\frac{1}{4}$.

2. (a) The potential energy function $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ can be used to derive a special case of the Duffing Equation

$$\ddot{x} - x + x^3 = 0$$

This second order, nonlinear differential equation is both undamped and unforced. Under the variable transformation $v = \dot{x} \implies \dot{v} = \ddot{x}$, substitute $v$ and $\dot{v}$ into the special case of the Duffing Equation and solve for $\dot{v}$ to obtain the system of equations

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = x - x^3$$

(See Appendix 1 for all details.)

(b) Recall that nullclines of a system of differential equations are obtained by restricting one direction of motion in the system:

$$\text{Let } \dot{x} = 0 \implies v(t) = 0$$

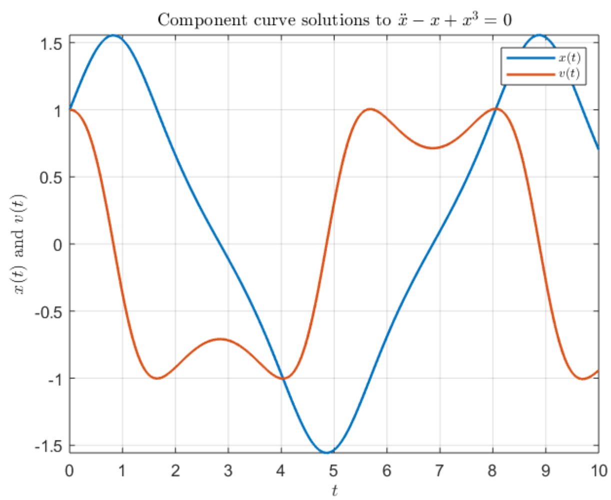
$$\text{Let } \dot{v} = 0 \implies x - x^3 = 0 \implies x = 0 \text{ or } x = \pm 1$$

The vertical nullcline is $v(t) = 0$, while the horizontal nullclines are $x(t) = 0$, $x(t) = \pm 1$. The intersections of vertical and horizontal nullclines give rise to the equilibrium solutions of the system, wherein both directions of motion are required to be 0. The system exhibits three equilibrium solutions of the form

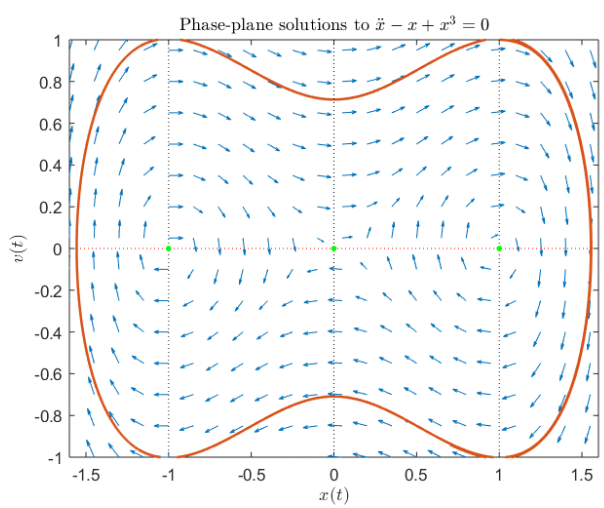
$$(x_{eq}, v_{eq}) = (0,0), \quad (x_{eq}, v_{eq}) = (1,0), \quad \text{and} \quad (x_{eq}, v_{eq}) = (-1,0)$$

(c) Letting the system start from the initial state $x(0) = v(0) = 1$ (so that both the position and potential energy of an arbitrary particle in the system are 1) the system of equations was solved using the Matlab solver `ode45` over the interval $t \in [0, 10]$.

i. The component curve solutions of the system are




ii. The phase-plane solution of the system (overlaid with the phase-portrait and the nullclines and equilibria of the system) is



From these plots, it appears that the $x(t)$ and $v(t)$ solutions appear to evolve periodically with time. From the phase-plane solution, we see that the parametric curve orbits the set of equilibrium solutions. The equilibrium $(0, 0)$ seems to be unstable while the other equilibria $(\pm 1, 0)$ appear to be centers.

# 3 Transition to Chaos: Forcing and Damping

We saw in the section above that the phase-space solutions conserved energy and resulted in curves that followed a single, exact trajectory without deviation. To eliminate conservation of energy (and to allow the potential for chaotic behavior in the Duffing system) two additional terms must be included in the system. A term, $\delta \dot{x}$, must be included to allow *damping* in the system and a term, $\gamma \cos(\omega t)$, must be included to allow *external forcing* of the system. The result of including these terms is the general Duffing Equation:

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$

1. This equation can be characterized as a second order, nonlinear oscillator with constant coefficients.

   There is periodic external forcing that comes from the term $\gamma \cos(\omega t)$. The parameter $\gamma$ is the strength of the driving force and $\omega$ is the frequency of forcing.

   The term $\alpha x$ is a classical restoring force that follows Hooke's Law (where $\alpha$ is a linear "stiffness" term that is equivalent to a classical spring constant). Meanwhile, the term $\beta x^3$ represents a cubic restoring force that controls the nonlinear response of the system. This often leads to an increase in the "stiffness" of the spring since it deviates from classical harmonic motion.

   The term $\delta \dot{x}$, $\delta \geq 0$, represents linear damping in the mass-spring system. (The term $\dot{x}$ is the velocity of the system.)

   The term $\ddot{x}$ is the acceleration of the system under the assumption that the system has mass $m = 1$.

2. In order to study the nonlinear Duffing Equation, it is useful to convert the differential equation to a system of first order differential equations.

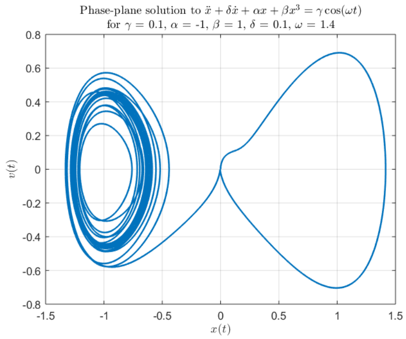
   Introduce the variable transformation $v = \dot{x} \implies \dot{v} = \ddot{x}$. Substituting these terms into the Duffing Equation and solving for the term $\dot{v}$ leads to the system of equations

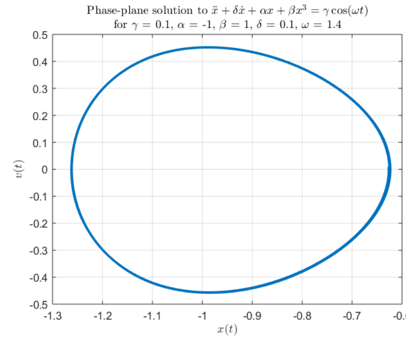
   $$\frac{dx}{dt} = v$$
   $$\frac{dv}{dt} = -\delta v - \alpha x - \beta x^3 + \gamma \cos(\omega t)$$

   (See Appendix 1 for all details.)

3. As discussed above, $\gamma$ represents the strength of the external driving force of the nonlinear system. Increases the driving force will push the system from deterministic dynamics to chaotic dynamics that cannot be predicted exactly. Investigating the behavior of the system as $\gamma$ increases yields the following results:

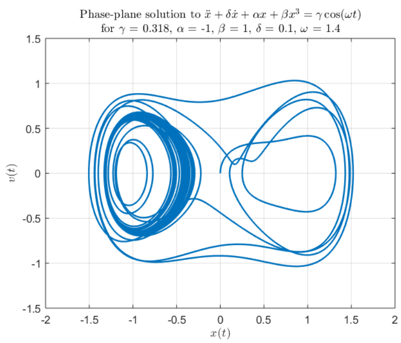
- Let $\gamma = 0.1$ and consider the range of time values $t \in [0, 200]$.

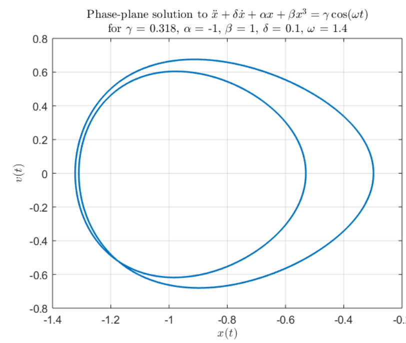



Plotting both the full parametric solution and the tail of the solution, we see that the full solution appears to exhibit very strange and unpredictable behavior. However, once the end behavior of the solution is examined more closely we see that the solution approaches a single orbit with period $T = 2\pi/\omega$.

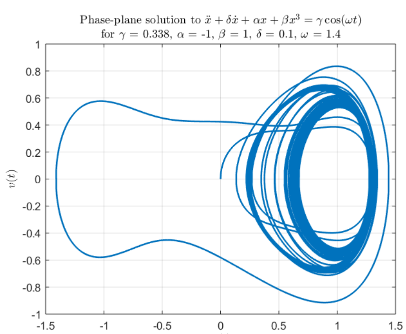
- Let $\gamma = 0.318$ and consider the range of time values $t \in [0, 800]$.

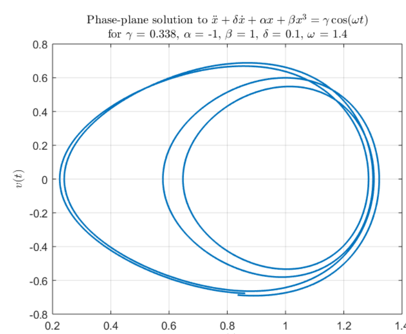



Increasing both the magnitude of the driving force and the interval $t$ over which the solution is computed, we see phenomena similar to those in the previous solution plots. The initial behavior of the solution is unpredictable but the end behavior of the solution approaches a simple curve that is composed of two nested orbits. This demonstrates "period doubling" in the solution (wherein the period of the solution is $T = 4\pi/\omega$).

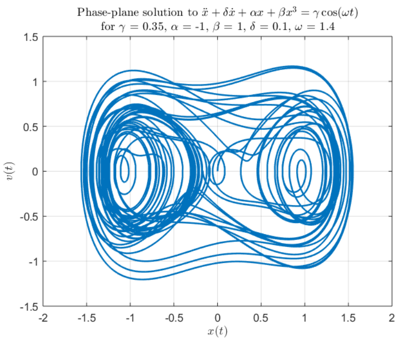
- Let $\gamma = 0.338$ and consider the range of time values $t \in [0, 2000]$.

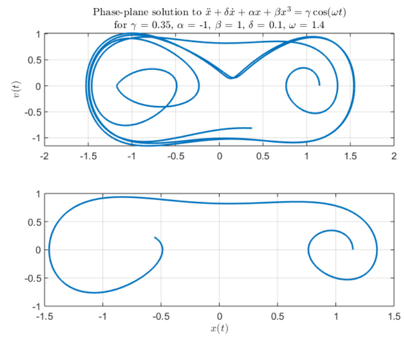



Further increasing the magnitude of the driving force and the length of the $t$ interval reveals the same pattern of behavior as in the previous two cases. In this case, the end behavior of the solution gives rise yet again to period doubling (as can be seen in the four nested orbits of the solution) and the period is $T = 8\pi/\omega$.

- Let $\gamma = 0.35$ and consider the range of time values $t \in [0, 3000]$.




This final increase in the driving force $\gamma$ reveals a very different type of behavior in the Duffing system. While the initial behavior of the other solutions appeared just as unpredictable as this final solution, the final behavior of the other solutions settled down to a single set of nested orbits that reveal bifurcation in the system. However, the final parameter values reveal that the system has transitioned from the phase space where bifurcation occurs into a region of chaos. This does not allow for the system to eventually reach a stable, fixed behavior but instead continue to move through the phase-space in an unpredictable fashion.

## 4 Conclusion

This report investigated the Duffing Equation for a range of parameter values. We found, due to energy conservation, that the Duffing Equation is unable to exhibit chaos in when the oscillator is undamped and unforced (that is, $\delta = \gamma = 0$). To allow for chaos, energy conservation is eliminated by including both a damping and an external forcing term. Then we see, as the magnitude of external forcing is increased, the system moves through a region of period-doubling bifurcations and then transitions to a chaotic regime. The transition to chaos appears to occur between $\gamma = 0.338$ and $\gamma = 0.35$.

# 5 Appendix 1

- Conversion of the Special Case Duffing Equation to a System of ODEs:

$$\text{Let } \dot{x} = v \implies \ddot{x} = \dot{v}$$
$$\ddot{x} - x + x^3 = 0$$
$$\implies \dot{v} - x + x^3 = 0$$
$$\implies \dot{v} = x + x^3$$

The system of equations is then

$$\begin{cases} \dot{x} = v \\ \dot{v} = x - x^3 \end{cases}$$

- Conversion of the Duffing Equation to a System of ODEs:

$$\text{Let } \dot{x} = v \implies \ddot{x} = \dot{v}$$
$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$
$$\implies \dot{v} + \delta v + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$
$$\implies \dot{v} = -\delta v - \alpha x - \beta x^3 + \gamma\cos(\omega t)$$

The system of equations is then

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\delta v - \alpha x - \beta x^3 + \gamma\cos(\omega t) \end{cases}$$

# 6 Appendix 2

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... Matlab code would go here! ...
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