APPM 3570/STAT 3100 — Exam 2 — Spring 2024

On the front of your bluebook, write (1) **your name**, (2) **Exam 2**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

> On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:______________________________ Date:__________

1. [EXAM02] (32pts) There are 4 unrelated parts to this question. Justify your answers.

   (a) (8pts) On a stretch of highway, the number of automobile accidents occur with a Poisson distribution at an average of three accidents per week. Calculate the probability that there are at most 2 accidents occurring in any given week on this stretch of the highway.

   (b) (8pts) In a small city, the number of automobiles accidents occur with a Poisson distribution at an average of three accidents per week. What is the probability that we have to *wait* at least two weeks between any 2 accidents?

   (c) (8pts) Find the probability $P(-13 \le X \le 19)$ where X is a Normal random variable with parameters $\mu = -1$ and $\sigma^2 = 16$. Give your answer in terms of Φ, the cumulative distribution function of the Standard Normal rv.

   (d) (8pts) Two candies are selected at random from a jar containing three M&M's©, two Reese's Pieces™ and four Smarties®. If X and Y are, respectively, the number of M&M's and Reese's Pieces included among the two candies drawn from the jar, find the *joint probability mass function* of X and Y. (Be sure to define the pmf for all real numbers.)

Solution:

(a)(8pts) Let X be the number of accidents per week, then $X \sim \text{Poisson}(\lambda = 3)$ with pmf $p(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, for $k = 0, 1, \ldots$, and we need to find $P(X \le 2)$. Note that

$$P(X \le 2) = p(0) + p(1) + p(2) = e^{-3}\Big(1 + 3 + \frac{9}{2!}\Big) = e^{-3}\frac{17}{2} \approx 0.4232.$$

(b)(8pts) Let T be the time (in weeks) between successive accidents. Since the number of accidents occurs with a Poisson distribution, the time between accidents follows the *exponential distribution*. If there are an average of three accidents per week then, on average, there is $E[T] = \frac{1}{3} = \frac{1}{\lambda}$ of a week between accidents and so $T \sim \text{Exp}(\lambda = 3)$ with pdf $f(x) = \lambda e^{-\lambda x}$. The probability that there are at least two weeks between any two accidents is

$$P(T \ge 2) = 1 - P(T < 2) = 1 - \int_0^2 3e^{-3x}\,dx = 1 - \Big(-e^{-3x}\Big|_0^2\Big) = 1 - \Big(1 - e^{-6}\Big) = e^{-6} \approx 0.0025.$$

(c)(8pts) Here $\mu = -1$ and $\sigma = \sqrt{16} = 4$, so standardizing X yields

$$P(-13 \le X \le 19) = P\left(\frac{-13 - (-1)}{4} \le \frac{X - \mu}{\sigma} \le \frac{19 - (-1)}{4}\right) = P\left(-3 \le Z \le 5\right) = \Phi(5) - \Phi(-3).$$

Note that, due to the symmetry of the pdf of Z around $z = 0$, we have that $\Phi(-3) = 1 - \Phi(3)$ so we could also write the probability as $P(-13 \le X \le 19) = \Phi(5) + \Phi(3) - 1$.

(d)(8pts) The possible pairs are $(X, Y) \in \{(0,0),(1,0),(0,1),(1,1),(2,0),(0,2)\}$. The size of the sample space of this experiment is $|S| = \binom{9}{2} = 36$, and, for example, to find $P(X = 1, Y = 0)$, we need to count the number of ways of selecting one M&M®, no Reese's Pieces© and one Smarties™, which can be done in $\binom{3}{1}\binom{2}{0}\binom{4}{1} = 12$ ways, so, $P(X = 1, Y = 0) = \frac{12}{36}$. In general, the pmf is

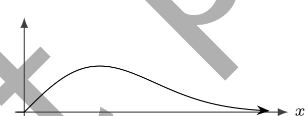
$$p(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}}{\binom{9}{2}} \quad \text{for } x = 0,1,2,\ y = 0,1,2 \text{ where } 0 \le x + y \le 2 \text{ and } 0 \text{ otherwise.}$$

We summarize the probabilities in the following table (where $P(X = i, Y = j) = 0$ for all other value of i, j):

Y \ X	0	1	2
0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
1	$\frac{2}{9}$	$\frac{1}{6}$	×
2	$\frac{1}{36}$	×	×

2. [Exam02] (40pts) *(Wind Turbines)* Let X denote the *vibratory stress level* (psi) on a wind turbine blade at a particular wind speed in a wind tunnel. Analysis of blade stress data collected by Department of Energy wind turbines at Rocky Flats suggests that the appropriate distribution model is the *Rayleigh distribution* with probability density function given by

$$f_X(x) = \begin{cases} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$



where $\theta > 0$ is called the *scale parameter* with $E[X] = \theta\sqrt{\frac{\pi}{2}}$ and $\text{Var}(X) = \frac{4-\pi}{2}\theta^2$.

(a) (10pts) Find F_X, the *cumulative distribution function* of X and then *verify* that $f_X(x)$ is a legitimate probability density function. (Be sure to define the cdf for all real numbers.)

(b) (10pts) Suppose $\theta = 10$, what is the probability that the vibratory stress level is between 10psi and 20psi?

(c) (10pts) Suppose $g(x) = 2x^2 + 20$ find $E[g(X)]$. *(Hint: No integrals need to be calculated to answer this part.)*

(d) (10pts) If $Y = 1 - e^{-X^2/(2\cdot 10^2)}$, find the *probability density function* of Y. *(Hint: Note that $0 < Y < 1$.)* Be sure to define the pdf for all real numbers.

Solution:

(a)(10pts) For any $a > 0$, if we let $u = \frac{x^2}{2\theta^2} \Rightarrow du = \frac{x}{\theta^2}dx$, then

$$F_X(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx = \int_0^a \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\,dx$$

$$= \int_0^{a^2/(2\theta^2)} e^{-u}\,du$$

$$= -e^{-u}\Big|_0^{a^2/(2\theta^2)} = 1 - e^{-\frac{a^2}{2\theta^2}} \Rightarrow F_X(a) = \begin{cases} 0, & \text{if } a \le 0 \\ 1 - e^{-\frac{a^2}{2\theta^2}}, & \text{if } a > 0. \end{cases}$$

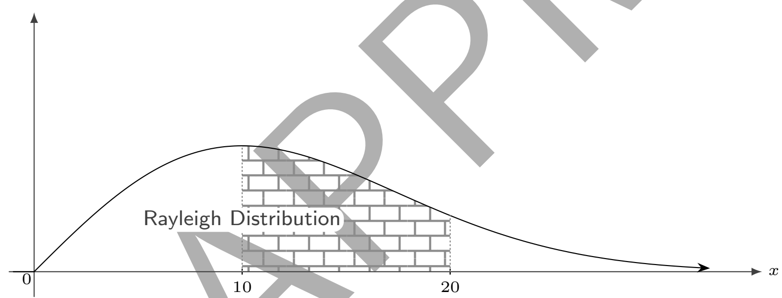
To verify the pdf, note that $f_X(x) \geq 0$ and,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\,dx = \lim_{a\to\infty} \int_0^{a} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\,dx = \lim_{a\to\infty} 1 - e^{-\frac{a^2}{2\theta^2}} = 1$$

so $f_X(x)$ is a legitimate pdf.

(b)(10pts) To find the probability of the event $\{10 \leq X \leq 20\}$, note that

$$P(10 \leq X \leq 20) = F_X(20) - F_X(10) = \left(1 - e^{-20^2/(2\cdot 10^2)}\right) - \left(1 - e^{-10^2/(2\cdot 10^2)}\right) = e^{-1/2} - e^{-2} \approx 0.4712.$$



(c)(10pts) The shortcut formula $\text{Var}(X) = E[X^2] - E[X]^2$ implies that $E[X^2] = \text{Var}(X) + E[X]^2$, thus

$$\begin{aligned} E[g(X)] = E[2X^2 + 20] &= 2E[X^2] + E[20] \\ &= 2\big(\text{Var}(X) + E[X]^2\big) + 20 \\ &= 2\Big(\frac{4-\pi}{2}\theta^2\Big) + 2\Big(\theta\sqrt{\frac{\pi}{2}}\Big)^2 + 20 = \theta^2(4-\pi) + \theta^2\pi + 20 = 4\theta^2 + 20 = 420\text{psi}. \end{aligned}$$

(d)(10pts) Since $X > 0$ and $\theta = 10$, we see that $Y = 1 - e^{-X^2/(2\cdot 10^2)} = 1 - e^{-X^2/(2\cdot\theta^2)} = F_X(X)$ and we wish to find $P(Y \leq a)$. Since $0 < Y < 1$, we have $P(Y \leq a) = 0$ if $a \leq 0$ and $P(Y \leq a) = 1$ if $a \geq 1$. Finally, if $a \in (0,1)$, we have

$$\begin{aligned} F_Y(a) &= P(Y \leq a) = P(F_X(X) \leq a) = P\big(X \leq F_X^{-1}(a)\big) = F_X\big(F_X^{-1}(a)\big) = a \\ \Rightarrow\ f_Y(a) &= \frac{d}{da} F_Y(a) = \begin{cases} 1, & \text{if } a \in (0,1), \\ 0, & \text{else}, \end{cases} \end{aligned}$$

so $Y \sim \text{Uniform}(0,1)$. (For $X > 0$, the function $Y = F_X(X)$ has an inverse since it is an increasing function.)

By Definition Alternately, for any $a \in (0,1)$, we have

$$\begin{aligned} F_Y(a) = P(Y \leq a) &= P(1 - e^{-X^2/(2\cdot 10^2)} \leq a) \\ &= P\big(-\tfrac{X^2}{2\cdot 10^2} \geq \ln(1-a)\big) \\ &= P\big(0 < X \leq \sqrt{-2\cdot 10^2 \ln(1-a)}\big) \\ &= \int_0^{\sqrt{-2\cdot 10^2 \ln(1-a)}} \frac{x}{10^2} e^{-\frac{x^2}{2\cdot 10^2}}\,dx \\ &= -e^{-\frac{x^2}{2\cdot 10^2}}\Big|_0^{\sqrt{-2\cdot 10^2\ln(1-a)}} = -(1-a) + 1 = a \ \Rightarrow\ f_Y(a) = \begin{cases} 1, & \text{if } a \in (0,1), \\ 0, & \text{else}. \end{cases} \end{aligned}$$

3. [EXAM02] (28pts) Suppose that X and Y are *jointly distributed random variables* with *joint probability density function* given by

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & \text{if } 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

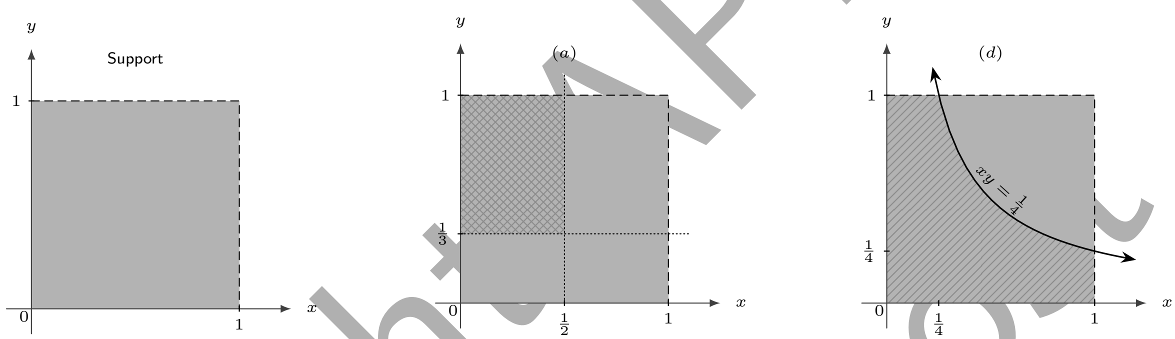
(a) (7pts) Set-up, but *do not solve* an integral (or integrals) to find $P(X < \frac{1}{2}, Y > \frac{1}{3})$.

(b) (7pts) Find the $f_X(x)$, *marginal probability density function* of X. (Be sure to define the pdf for all real numbers.)

(c) (7pts) Find the expectation $E[X]$.

(d) (7pts) Set-up, but *do not solve* an integral (or integrals) to find $P(XY < \frac{1}{4})$.

Solution:



(a)(7pts) Note that, using the joint pdf of (X,Y), we have

$$P(X < \tfrac{1}{2}, Y > \tfrac{1}{3}) = P(0 < X < \tfrac{1}{2}, \tfrac{1}{3} < Y < 1) = \int_{1/3}^{1}\int_{0}^{1/2} \frac{2}{5}(2x+3y)\,dxdy = \int_{0}^{1/2}\int_{1/3}^{1} \frac{2}{5}(2x+3y)\,dydx.$$

(b)(7pts) For each $x \in (0,1)$, the marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{2}{5}(2x+3y)\,dy = \frac{2}{5}\Big(2xy + \frac{3y^2}{2}\Big|_{y=0}^{y=1}\Big) = \frac{2}{5}\Big(2x + \frac{3}{2}\Big) \text{ for } x \in (0,1)$$

and 0 otherwise.

(c)(7pts) Using the marginal density $f_X(x)$, we have

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot \frac{2}{5}\Big(2x + \frac{3}{2}\Big)\,dx$$
$$= \frac{2}{5}\Big(\frac{2x^3}{3} + \frac{3x^2}{4}\Big)\Big|_0^1 = \frac{2}{5}\Big(\frac{2}{3} + \frac{3}{4}\Big) = \frac{2}{5}\Big(\frac{17}{12}\Big) = \frac{17}{30}.$$

(d)(7pts) Note that (see graph)

$$\{XY < \tfrac{1}{4}\} = \{Y < \tfrac{1}{4X}\} = \{0 < X < \tfrac{1}{4},\ 0 < Y < 1\} \cup \{\tfrac{1}{4} < X < 1,\ 0 < Y < \tfrac{1}{4X}\},$$

so we have

$$P(XY < \tfrac{1}{4}) = \int_0^{1/4}\int_0^1 \frac{2}{5}(2x+3y)\,dydx + \int_{1/4}^{1}\int_0^{1/4x} \frac{2}{5}(2x+3y)\,dydx.$$

For a $dxdy$-integral, we have

$$P(XY < \tfrac{1}{4}) = \int_0^{1/4}\int_0^1 \frac{2}{5}(2x+3y)\,dxdy + \int_{1/4}^{1}\int_0^{1/4y} \frac{2}{5}(2x+3y)\,dxdy.$$