APPM 3570/STAT 3100 — Exam 1 — Spring 2024

On the front of your bluebook, write (1) **your name**, (2) **Exam 1**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers. Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.
Signature:____________________________ Date:_________

1. (40pts) There are 4 unrelated parts to this question. Justify your answers.

   (a) (10pts) How many different permutations of the letters in the word `T A L L A H A S S E E` are there?

   (b) (10pts) How many of the permutations of the letters in the word `T A L L A H A S S E E` have no adjacent `A`'s?

   (c) (10pts) Assuming we have a standard 52 card deck, how many ways can we select a five-card poker hand that contains exactly *two pairs* (that is, two of each of two different kinds and a fifth card of a third kind)?

   (d) (10pts) You pick one card from a standard 52 card deck. If E, F and G denote the events

   E : The card is a spade, F : The card is red, G : The card is a picture card (*i.e.*, a J, Q or K).

   Find $\mathrm{P}(E \cup F \cup G)$ using the Inclusion-Exclusion Principle. *Simplify your answer.*

Solution:

(a)(10pts) The word `TALLAHASSEE` has 11 letters, with three `A`'s, and two `L`'s, `S`'s and `E`'s, thus, adjusting for overcount, we have

$$\binom{11}{3,2,2,2,1,1} = \frac{11!}{3!2!2!2!1!1!} = 831{,}600 \text{ permutations of the letters in the word TALLAHASSEE.}$$

(b)(10pts) Do this as two experiments. First disregard the `A`'s, then, for the eight letters `TLLHSSEE`, we have $\frac{8!}{2!2!2!1!1!}$ possible permutations. Second, note that for any permutation of these eight letters, there are only nine possible positions for the three `A`'s so that they are not adjacent, for example: $_\wedge \mathtt{T}_\wedge \mathtt{L}_\wedge \mathtt{L}_\wedge \mathtt{H}_\wedge \mathtt{S}_\wedge \mathtt{S}_\wedge \mathtt{E}_\wedge \mathtt{E}_\wedge$.

So, for each of the $\frac{8!}{2!2!2!1!1!}$ permutations there are $\binom{9}{3}$ ways to select positions for the `A`'s, thus, by the product rule of counting, the number of permutations of the letters in the word `TALLAHASSEE` that have no adjacent `A`'s is

$$\frac{8!}{2!2!2!1!1!} \cdot \binom{9}{3} = 5040 \cdot 84 = 423{,}360.$$

(c)(10pts) We choose two ranks to be the pairs *and* then choose the suits for each of the pairs *and* then choose the fifth card (rank *and* suit) so we have

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1} = 123{,}552 \text{ ways to select a five-card poker hand that contains exactly two pairs.}$$

Caution: The count $\binom{13}{1}\binom{12}{1}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1} = 13 \cdot 12 \cdot \binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$ *overcounts* the answer by a factor of $2!$.

(d)(10pts) If E, F and G denote the events

E : The card is a spade, F : The card is red, G : The card is a picture card (*i.e.*, a J, Q or K),

then, by the Inclusion-Exclusion Principle, we have

$$\begin{aligned}
\mathrm{P}(E \cup F \cup G) &= \mathrm{P}(E) + \mathrm{P}(F) + \mathrm{P}(G) - \mathrm{P}(E \cap F) - \mathrm{P}(E \cap G) - \mathrm{P}(F \cap G) + \mathrm{P}(E \cap F \cap G) \\
&= \frac{13}{52} + \frac{26}{52} + \frac{12}{52} - \frac{0}{52} - \frac{3}{52} - \frac{6}{52} + \frac{0}{52} \\
&= \frac{42}{52} = \frac{21}{26} \approx 0.81.
\end{aligned}$$

2. (32pts) Ralphie can get to work in three different ways: by automobile, bus, or bicycle. On any given day that she has to go to work there is a 50% chance that she will be late when she drives her automobile. If she takes the bus, which uses a special lane reserved for buses, there is a 20% chance that she will be late. The probability that she is late when she rides her bicycle is only 10%.

   (a) (8pts) Chip has been keeping records. Chip knows that Ralphie drives 30% of the time, takes the bus only 10% of the time, and takes her bicycle 60% of the time. Chip wants to know what is the probability that Ralphie is late?

   (b) (8pts) Ralphie is late today, Chip wants to know what is the probability that Ralphie drove her automobile?

   (c) (8pts) Chip discovers that Ralphie has decided to take the bus to work for the next five days of work. Chip wants to know what is the probability that Ralphie will be late less than two times in those five days of work? *State any assumptions you are making.*

   (d) (8pts) Let A be the event that Ralphie drives her automobile to work and let L be the event that Ralphie is late, find the conditional probability $P(L \mid A^c)$.

Solution:

(a)(8pts) Let A be the event that Ralphie drives her automobile, B is the event she takes the bus and let C be the event that she rides her bicycle. Let L be the event that Ralphie is late, we are given that

$$P(L \mid A) = \frac{5}{10}, \qquad P(L \mid B) = \frac{2}{10}, \qquad P(L \mid C) = \frac{1}{10}.$$

Note that $L = (L \cap A) \cup (L \cap B) \cup (L \cap C)$ and, if we assume $P(A) = \frac{3}{10}$, $P(B) = \frac{1}{10}$ and $P(C) = \frac{6}{10}$, then, by conditioning, we have

$$\begin{aligned} P(L) = P\big(LA \cup LB \cup LC\big) &= P(LA) + P(LB) + P(LC) \\ &= P(L \mid A)P(A) + P(L \mid B)P(B) + P(L \mid C)P(C) \\ &= \frac{5}{10}\cdot\frac{3}{10} + \frac{2}{10}\cdot\frac{1}{10} + \frac{1}{10}\cdot\frac{6}{10} = \frac{15+2+6}{100} = \frac{23}{100}. \end{aligned}$$

Thus, based on Chip's records, there is a 23% chance Ralphie will be late.

(b)(8pts) We need to find $P(A \mid L)$, note that (by Bayes' Rule) we have

$$P(A \mid L) = \frac{P(A \cap L)}{P(L)} = \frac{P(L \mid A)P(A)}{P(L \mid A)P(A) + P(L \mid B)P(B) + P(L \mid C)P(C)} = \frac{\frac{5}{10}\cdot\frac{3}{10}}{\frac{23}{100}} = \frac{15}{23} \approx 0.65.$$

So, if Ralphie is late, then there is a $\frac{15}{23}$ probability that she drove her automobile to work.

(c)(pts) Let $p = P(L \mid B) = \frac{2}{10}$ then $P(L^c \mid B) = 1 - p = \frac{8}{10}$ and Ralphie can be late k times out of five in $\binom{5}{k}$ ways, with each occurrence having probability $p^k(1-p)^{5-k}$ (assuming Ralphie's commute on each day is **independent** of the other), thus

$$\begin{aligned} P\left(\begin{smallmatrix}\text{Ralphie is late}\\ \text{less than 2 times}\end{smallmatrix}\right) &= P\left(\left\{\begin{smallmatrix}\text{Ralphie is late}\\ \text{zero times}\end{smallmatrix}\right\} \cup \left\{\begin{smallmatrix}\text{Ralphie is late}\\ \text{exactly one time}\end{smallmatrix}\right\}\right) \\ &= \binom{5}{0}(1-p)^5 + \binom{5}{1}p(1-p)^4 = \left(\frac{8}{10}\right)^5 + 5\left(\frac{2}{10}\right)\left(\frac{8}{10}\right)^4 = (0.8^4)\cdot(1.8) \approx 0.74. \end{aligned}$$

(d)(8pts) We find $P(L \mid A^c)$, the probability that Ralphie is late given that she does *not* drive her automobile to work. Keeping in mind that $A^c = B \cup C$, we have

$$\begin{aligned} P(L \mid A^c) = P(L \mid B \cup C) = \frac{P(L \cap (B \cup C))}{P(B \cup C)} &= \frac{P(LB \cup LC)}{P(B \cup C)} \\ &= \frac{P(LB) + P(LC)}{P(B) + P(C)} \\ &= \frac{P(L \mid B)P(B) + P(L \mid C)P(C)}{P(B) + P(C)} = \frac{\frac{2}{10}\cdot\frac{1}{10} + \frac{1}{10}\cdot\frac{6}{10}}{\frac{1}{10} + \frac{6}{10}} = \frac{8}{70} \approx 0.11. \end{aligned}$$

Thus, if Ralphie does not use her automobile to get to work, then then the likelihood she will be late is $\frac{4}{35}$.

Alternately, note that, using the fact that $L = (L \cap A) \cup (L \cap A^c)$, we have

$$P(L \mid A^c) = \frac{P(L \cap A^c)}{P(A^c)} = \frac{P(L) - P(L \cap A)}{P(A^c)} = \frac{P(L) - P(L \mid A)P(A)}{1 - P(A)} = \frac{\frac{23}{100} - \frac{5}{10} \cdot \frac{3}{10}}{1 - \frac{30}{100}} = \frac{8}{70} = \frac{4}{35}.$$

3. (28pts) Suppose that a fair, six-sided die is rolled twice. Let the random variable Y denote the value of the first roll minus the value of the second roll.

   (a) (7pts) What are all the possible values that the random variable Y can take on with positive probability?

   (b) (i)(5pts) Find the *probability mass function* (pmf) of Y. (Your pmf should be defined for all real numbers.)
   (ii)(2pts) Verify that your answer is a true probability mass function.

   (c) (7pts) Find the probability that $Y^2 + Y = 2$.

   (d) (7pts) If Y is even, what is the probability that the first roll is a 1? (The even integers are defined to be $0, \pm 2, \pm 4, \ldots$.)

Solution:

(a)(7pts) Note that $Y =$ Roll #1 − Roll #2 and the largest Roll #1 can be is a "6" and the smallest Roll #2 can be is a "1" so the largest Y can be is $6 - 1 = 5$ and similarly the smallest Y can be is $1 - 6 = -5$ and since the value of each roll can increment by one and since Y is a discrete random variable we see that the possible values that Y can take on are

$$Y \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}.$$

(b)(i)(5pts) Now, using the following table of differences between Roll #1 and Roll #2 we can find the probability mass function of Y

Roll 1 \ Roll 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Roll #1 minus Roll #2

Roll 1 \ Roll 2	1	2	3	4	5	6
1	0	−1	−2	−3	−4	−5
2	1	0	−1	−2	−3	−4
3	2	1	0	−1	−2	−3
4	3	2	1	0	−1	−2
5	4	3	2	1	0	−1
6	5	4	3	2	1	0

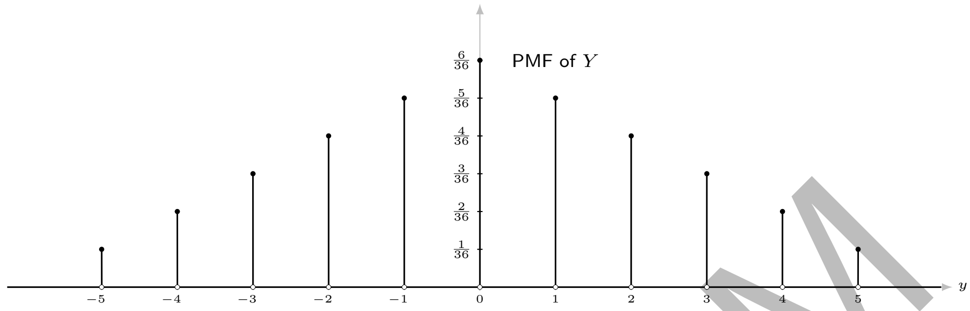
so we see that the pmf of Y can be listed as follows:

$$\begin{aligned} &p_Y(1) = p_Y(-1) = \tfrac{5}{36}, \quad p_Y(2) = p_Y(-2) = \tfrac{4}{36}, \\ &p_Y(3) = p_Y(-3) = \tfrac{3}{36}, \quad p_Y(4) = p_Y(-4) = \tfrac{2}{36}, \\ &p_Y(5) = p_Y(-5) = \tfrac{1}{36}, \quad p_Y(0) = \tfrac{6}{36}, \text{ and otherwise } p_Y(k) = 0. \end{aligned}$$

Alternately, we could simply write $p_Y(y) = \begin{cases} \frac{6-|y|}{36}, & \text{for } y \in \{-5, -4, \ldots, 4, 5\}, \\ 0, & \text{otherwise.} \end{cases}$

(b)(ii)(2pts) To verify we sum up all the pmf values given above, note that

$$\sum_{k \in \mathbb{R}} p_Y(k) = 2\left(\frac{5}{36}\right) + 2\left(\frac{4}{36}\right) + 2\left(\frac{3}{36}\right) + 2\left(\frac{2}{36}\right) + 2\left(\frac{1}{36}\right) + \frac{6}{36} = \frac{10 + 8 + 6 + 4 + 2 + 6}{36} = \frac{36}{36} = 1.$$



(c)(7pts) Note that

$$Y^2 + Y = 2 \;\Rightarrow\; Y^2 + Y - 2 = 0 \;\Rightarrow\; (Y-1)(Y+2) = 0 \;\Rightarrow\; Y = 1 \text{ or } Y = -2$$

so

$$P(Y^2 + Y = 2) = P(\{Y = 1\} \cup \{Y = -2\}) = P(Y = 1) + P(Y = -2) = \frac{5}{36} + \frac{4}{36} = \frac{9}{36} = \frac{1}{4}$$

where we used the fact that $\{Y = 1\} \cap \{Y = -2\} = \emptyset$.

(d)(7pts) We need to find the conditional probability $P(\text{Roll \#1=1} \mid Y \text{ is even})$. First note that

$$P(Y \text{ is even}) = P(Y \in \{-4, -2, 0, 2, 4\}) = \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36} = \frac{18}{36} = \frac{1}{2}$$

and

$$P(\{\text{Roll \#1=1}\} \cap \{Y \text{ is even}\}) = P(\{(1,1), (1,3), (1,5)\}) = \frac{3}{36} = \frac{1}{12}$$

so the probability that the first roll is a 1 given that Y is even is

$$P(\text{Roll \#1=1} \mid Y \text{ is even}) = \frac{P(\{\text{Roll \#1=1}\} \cap \{Y \text{ is even}\})}{P(Y \text{ is even})} = \frac{1/12}{1/2} = \frac{2}{12} = \frac{1}{6}.$$