APPM 3570/STAT 3100 - Section 002 Exam 3 (Final) Spring 2023

NAME: ______________________________

SECTION: 001 *or* 002

Instructions:

1. Calculators are permitted.
2. Notes, your text and other books, cell phones, and other electronic devices are not permitted—except for calculators or as needed to view and upload your work.
3. Justify your answers, show all work.
4. A table summarizing the main probability distributions discussed in the course, and another one with tabular values for the c.d.f. of a standard Normal distribution, may be found at the end of the exam.
5. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
6. Don't forget to scan any back pages you used for extra space!
7. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
8. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________ Date: ____________

Duration: 90 minutes

Problem 1. (20 points.) A professional soccer player scores a random number of goals each day. On those days when he is in a bad mood, this number is a Poisson random variable with mean 1. On those days when he is in a good mood, the number of goals is a Poisson random variable with mean 2. Assume that he is in a good mood on any given day with probability $\frac{1}{4}$ and otherwise he is in a bad mood.

(a) What's the expected number of goals he will score tomorrow?

(b) Yesterday, he scored one goal. What is the conditional probability that he was in a good mood?

Solution:

(a) (10 points.) Let G denote the number of goals he makes in a day.
Let $M = 1$ if the player is in a good mood on a given day and $M = 0$ if the player is in a bad mood on a given day.

$G|M = 1 \sim Poisson(\lambda = 2)$
$\mathrm{E}[G|M = 1] = 2$

$G|M = 0 \sim Poisson(\lambda = 1)$
$\mathrm{E}[G|M = 0] = 1$

$$\begin{aligned}\mathrm{E}[G] &= \mathrm{E}\Big[\mathrm{E}[G|M]\Big]\\ &= \sum_m \mathrm{E}[G|M = m]\,\mathrm{P}(M = m)\\ &= \mathrm{E}[G|M = 1]\,\mathrm{P}(M = 1) + \mathrm{E}[G|M = 0]\,\mathrm{P}(M = 0)\\ &= (2)\left(\frac{1}{4}\right) + (1)\left(\frac{3}{4}\right)\\ &= \frac{5}{4}\end{aligned}$$

(b) (10 points.)

$$\begin{aligned}\mathrm{P}(M = 1|G = 1) &= \frac{\mathrm{P}(M = 1, G = 1)}{\mathrm{P}(G = 1)}\\ &= \frac{\mathrm{P}(G = 1|M = 1)\,\mathrm{P}(M = 1)}{\mathrm{P}(G = 1, M = 1)\,\mathrm{P}(M = 1) + \mathrm{P}(G = 1, M = 0)\,\mathrm{P}(M = 0)}\\ &= \frac{2e^{-2}\left(\frac{1}{4}\right)}{2e^{-2}\left(\frac{1}{4}\right) + e^{-1}\left(\frac{3}{4}\right)}\\ &\approx .197\end{aligned}$$

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Problem 2. (35 points.) Let X and Y be independent random variables, each of which is exponential with parameter λ. Consider the random variables $V = \frac{X}{Y}$ and $W = X + Y$.

(a) What's the joint p.d.f. of X and Y?

(b) Determine X and Y as functions of V and W.

(c) What's the joint p.d.f. of V and W?

(d) Find the marginal p.d.f. of V and the marginal p.d.f. of W.

(e) Are V and W independent? Justify your answer.

Solution:

(a) (8 points.)

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2\, e^{-\lambda(x+y)} & x > 0, \quad y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) (8 points.) $X = \frac{VW}{V+1}$ and $Y = \frac{W}{V+1}$

(c) (8 points.) $h_1(v,w) = \frac{vw}{v+1}$ and $h_2(v,w) = \frac{w}{v+1}$

$$J(v,w) = \begin{bmatrix} \frac{\partial h_1}{\partial v} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial v} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{w}{(v+1)^2} & \frac{v}{v+1} \\ \frac{-w}{(v+1)^2} & \frac{1}{v+1} \end{bmatrix} = \frac{w}{(v+1)^2}$$

$|J(v,w)| = \left|\frac{w}{(v+1)^2}\right| = \frac{w}{(v+1)^2}$ since $w > 0$.

$$\begin{aligned} f_{V,W}(v,w) &= \lambda^2\, e^{-\lambda(\frac{vw}{v+1} + \frac{w}{v+1})}\left(\frac{w}{(v+1)^2}\right) \\ &= \frac{\lambda^2 w}{(v+1)^2} e^{-\lambda w} \quad v > 0, \quad w > 0 \end{aligned}$$

(d) (8 points.)

$$\begin{aligned} f_V(v) &= \int_0^\infty \frac{\lambda^2 w}{(v+1)^2} e^{-\lambda w}\, dw \quad \text{where } \lambda > 0 \\ &= \frac{\lambda^2}{(v+1)^2} \int_0^\infty w e^{-\lambda w}\, dw \\ &= \frac{\lambda^2}{(v+1)^2}\left(-\frac{w}{\lambda} e^{-\lambda w}\Big|_0^\infty + \int_0^\infty \frac{1}{\lambda} e^{-\lambda w}\, dw\right) \\ &= \frac{\lambda^2}{(v+1)^2}\left(\frac{1}{\lambda^2}\right) \\ &= \frac{1}{(v+1)^2} \end{aligned}$$

$$f_V(v) = \begin{cases} \frac{1}{(v+1)^2} & v > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_W(w) &= \int_0^\infty \frac{\lambda^2 w}{(v+1)^2} e^{-\lambda w}\, dv \quad \text{where } \lambda > 0 \\ &= \lambda^2 w e^{-\lambda w} \int_0^\infty \frac{1}{(v+1)^2}\, dv \end{aligned}$$

$$
\begin{aligned}
&= \lambda^2 w e^{-\lambda w} \int_1^\infty \frac{1}{u^2}\, du \\
&= \lambda^2 w e^{-\lambda w} \left(-\frac{1}{u} \Big|_0^\infty \right) \\
&= \lambda^2 w e^{-\lambda w}
\end{aligned}
$$

$$
f_W(w) = \begin{cases} \lambda^2 w e^{-\lambda w} & w > 0 \\ 0 & \text{otherwise} \end{cases}
$$

(e) (3 points.) Yes, V and W are independent. One can see that the joint p.d.f. can be factored into a function of v and a function of w for all (v, w).

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Problem 3. (30 points.) Suppose that X and Y are jointly continuous random variables and the conditional probability density function of X given Y is

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{1-y} & 0 < x < 1-y \\ 0 & \text{otherwise} \end{cases}$$

when $0 < y < 1$.

Suppose the marginal distribution of Y is

$$f_Y(y) = \begin{cases} 2(1-y) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\mathrm{E}[X|Y]$

(b) Find $\mathrm{E}[X]$

(c) Let $Y_1, Y_2, Y_3, \ldots$ be a random sample from the population with p.d.f. $f_Y(y)$. Find $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i^2$.

Solution:

(a) (10 points.)

$$\begin{aligned} \mathrm{E}[X|Y] &= \int_0^{1-y} x f_{X|Y}(x|y)\, dx \\ &= \int_0^{1-y} x\left(\frac{1}{1-y}\right) dx \\ &= \frac{x^2}{2(1-y)}\Big|_0^{1-y} \\ &= \frac{1-y}{2} \end{aligned}$$

(b) (10 points.) **Solution I.**

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}\Big[\mathrm{E}[X|Y]\Big] \\ &= \int_0^1 \mathrm{E}[X|Y]\, f_Y(y)\, dy \\ &= \int_0^1 \frac{1-y}{2}\, 2(1-y)\, dy \\ &= \int_0^1 (1-2y+y^2)\, dy \\ &= \left(y - y^2 + \frac{y^3}{3}\right)\Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

Solution II.

$$\mathrm{E}[X] = \int_0^1 x\, f_X(x)\, dx$$

$$
\begin{aligned}
&= \int_0^1 x \int_0^{1-x} 2\, dy\, dx \\
&= \int_0^1 2x(1-x)\, dx \\
&= \frac{1}{3}
\end{aligned}
$$

(c) (10 points.)

$$
\begin{aligned}
\mathrm{E}[Y_i^2] &= \int_0^1 y^2\, f_Y(y)\, dy \\
&= \int_0^1 y^2\, 2(1-y)\, dy \\
&= \int_0^1 2(y^2 - y^3)\, dy \\
&= 2\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Big|_0^1 \\
&= 2\left(\frac{1}{3} - \frac{1}{4}\right) \\
&= \frac{1}{6}
\end{aligned}
$$

Since the Y_i are *iid* and $\mathrm{E}[Y_i^2]$ is finite, the LLN implies that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i^2 = \mathrm{E}[Y_i^2] = \frac{1}{6}$.

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Problem 4. (15 points.)

You have invited 64 guests to a party. You need to make sandwiches for the guests. It is known that a guest will want 0, 1 or 2 sandwiches with probabilities $\frac{1}{4}, \frac{1}{2}$, and $\frac{1}{4}$, respectively. You assume that the number of sandwiches each guest wants is independent from other guests. You decide to make 75 sandwiches. What is the probability that you have made enough sandwiches? Justify any assumptions made.

Solution: Let X_i be the number of sandwiches wanted by guest i, $i = 1, \ldots, 64$. We find $\mathrm{E}[X_i] = (0)(\frac{1}{4}) + (1)(\frac{1}{2}) + (2)(\frac{1}{4}) = 1$.
We can find $\mathrm{Var}(X_i) = \mathrm{E}[X_i^2] - (\mathrm{E}[X_i])^2$.

$\mathrm{E}[X_i^2] = (0)(\frac{1}{4}) + (1)(\frac{1}{2}) + (4)(\frac{1}{4}) = \frac{3}{2}$.
Therefore, $\mathrm{Var}(X_i) = \frac{3}{2} - 1^2 = \frac{1}{2}$.

Since it is given the number of sandwiches wanted by a guest has a certain distribution, then we know the X_i are identically distributed, and it is given that a guest's desire for sandwiches is independent of other guests, so the X_i are *iid*.
Since $n = 64$ is large, we can employ the Central Limit Theorem. By the CLT,

$$\sum_{i=1}^{64} X_i \overset{(approx)}{\sim} \mathcal{N}(n\mu, n\sigma^2)$$
$$\overset{(approx)}{\sim} \mathcal{N}\left((64)(1),\, (64)(\frac{1}{2})\right)$$
$$\overset{(approx)}{\sim} \mathcal{N}(64,\, 32)$$

$\mathrm{P}(\sum_{i=1}^{64} X_i \le 75) \approx \mathrm{P}(Z \le \frac{75-64}{\sqrt{32}}) = \mathrm{P}(Z \le 1.9445) \approx .9738$.

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Bonus Problem. (Recover up to 4 points marked down in problems 1-4.) A miner is trapped in a mine containing 3 doors.

- The first door leads to a tunnel that returns her to the mine after 2 hours.
- The second door leads to a tunnel that will take her to safety after a number of hours that is Binomial with parameters $(4, \frac{1}{3})$.
- The third door leads to a tunnel that will take her to safety after 1 hour.

At all times, she is equally likely to choose any one of the doors. What is the expected number of hours until this miner reaches safety?

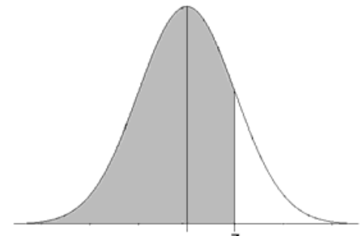
Solution: Let X denote the amount of time in hours until the miner reaches safety.
Let D denote the door she initially chooses.

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\Big[\mathrm{E}[X|D]\Big] \\
&= \sum_{d} \mathrm{E}[X|D=d]\,\mathrm{P}(D=d) \\
&= \mathrm{E}[X|D=1]\,\mathrm{P}(D=1) + \mathrm{E}[X|D=2]\,\mathrm{P}(D=2) + \mathrm{E}[X|D=3]\,\mathrm{P}(D=3) \\
&= (\mathrm{E}[X]+2)\left(\frac{1}{3}\right) + \left(\frac{4}{3}\right)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) \\
&= \frac{1}{3}\mathrm{E}[X] + \frac{13}{9} \\
\frac{2}{3}\mathrm{E}[X] &= \frac{13}{9} \\
\mathrm{E}[X] &= \frac{13}{6} \text{ hours}
\end{aligned}$$

Main Probability Distributions in APPM 3570/STAT 3100

Notation	pmf or pdf	Mean	Variance
$X \sim \text{Bernoulli}(p)$	$P(X=1)=p,\ P(X=0)=1-p$	p	$p(1-p)$
$X \sim \text{Binomial}(n,p)$	$P(X=k)=\binom{n}{k}\,p^k\,(1-p)^{n-k},\ k=0,\dots,n$	np	$np(1-p)$
$X \sim \text{Geometric}(p)$	$P(X=k)=(1-p)^{k-1}\,p,\ k=1,2,\dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$X \sim \text{NegBinomial}(r,p)$	$P(X=k)=\binom{k-1}{r-1}\,p^{k-r}\,p^r,\ k=r,r+1,\dots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
$X \sim \text{Poisson}(\lambda)$	$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!},\ k=0,1,\dots$	λ	λ
$X \sim \text{Uniform}(a,b)$	$f(x)=\frac{1}{b-a},\ a<x<b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$X \sim \text{Exponential}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x},\ x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$X \sim \text{Gamma}(k,\lambda)$	$f(x)=\frac{\lambda^k x^{k-1}}{(k-1)!}e^{-\lambda x},\ x>0$	$\frac{k}{\lambda}$	$\frac{k}{\lambda^2}$
$X \sim \text{Normal}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x\in\mathbb{R}$	μ	σ^2

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998