APPM 3570/STAT 3100 Exam 2 Spring 2023

NAME: ______________________________

SECTION: 001 or 002

Instructions:

1. Calculators are permitted.
2. Notes, your text and other books, cell phones, and other electronic devices are not permitted—except for calculators or as needed to view and upload your work.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5. Don't forget to scan any back pages you used for extra space!
6. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
7. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________ Date: ____________

Duration: 90 minutes

Problem 1. (24 points.) There are three unrelated parts to this question.

(a) Let X be a random variable such that $\mathrm{P}(X=1)=1-\mathrm{P}(X=0)>0$. If $5\,\mathrm{Var}(X)=\mathrm{E}(X)$, find $\mathrm{P}(X=0)$.

(b) Let U and V be discrete random variables with joint probability mass function (p.m.f.) given by the following table. What's the probability that $V=U^2$?

	$V=-1$	$V=0$	$V=1$
$U=-1$	5/38	1/19	3/19
$U=0$	1/38	3/19	1/19
$U=1$	7/38	4/19	1/38

(c) Let $Y\sim\mathrm{Geometric}(p=2/3)$. What's the expected value of 3^{1-2Y}?

Solution:

(a) (8 points.) Let $p=\mathrm{P}(X=1)$. Then $E(X)=p$ and $E(X^2)=E(X)=p$, therefore $\mathrm{Var}(X)=p-p^2$. If $5\,\mathrm{Var}(X)=E(X)$, then $5(p-p^2)=p$, which implies $p=\frac{4}{5}$ because $p>0$. So $\mathrm{P}(X=0)=1-p=\frac{1}{5}$.

(b) (8 points.)

$$P(V=U^2)=P(U=-1,V=1)+P(U=0,V=0)+P(U=1,V=1)$$
$$=\frac{3}{19}+\frac{3}{19}+\frac{1}{38}=\frac{13}{38}.$$

(c) (8 points.)

$$E(3^{1-2Y})=\sum_{k=1}^{\infty}3^{1-2k}\cdot\left(1-\frac{2}{3}\right)^{k-1}\frac{2}{3}=6\sum_{k=1}^{\infty}\left(\frac{1}{27}\right)^{k}=6\frac{1/27}{1-1/27}=\frac{6}{26}=\frac{3}{13}.$$

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Problem 2. (24 points.) There are three unrelated parts to this question.

(a) Let X be a random variable with cumulative distribution function (c.d.f.):

$$F(x) = \begin{cases} 0 & , x < -\ln(3); \\ \frac{3e^x - 1}{3(e^x+1)} & , x \geq -\ln(3). \end{cases}$$

Is X discrete, continuous, or neither? If discrete, determine its p.m.f. If continuous, determine its probability density function (p.d.f.).

(b) The life L, in years, of a certain type of electrical switch has an exponential distribution with an average life of 2 years. What is the probability it fails during the first year?

(c) Let $Y \sim \text{Normal}(16, 16)$. Find the expected value of $\frac{Y^2-16}{4}$.

Solution:

(a) (8 points.) Clearly, $F(x)$ is continuous for $x < -\ln(3)$ and $x > -\ln(3)$. On the other hand, since $3e^x = 1$ when $x = -\ln(3)$, we find that

$$F\big(-\ln(3)\big) = \frac{1-1}{3(1/3+1)} = 0 = \lim_{x \to -\ln(3)^-} F(x),$$

thus F is continuous everywhere, and X is a continuous random variable. Further, for $x > -\ln(3)$:

$$f(x) = \frac{d}{dx}\left[\frac{3e^x - 1}{3(e^x+1)}\right] = \frac{3e^x(e^x+1) - (3e^x-1)e^x}{3(e^x+1)^2} = \frac{4e^x}{3(e^x+1)^2}.$$

(b) (8 points.) Since $L \sim \text{Exponential}(\lambda = 1/2)$:

$$\text{P}(L < 1) = \int_0^1 \frac{1}{2} e^{-\frac{x}{2}}\, dx = -e^{-\frac{x}{2}}\Big|_0^1 = 1 - e^{-\frac{1}{2}} \, (\approx 0.393).$$

(c) (8 points.) Recall that $V(Y) = E(Y^2) - (EY)^2$. So:

$$E\left(\frac{Y^2-16}{4}\right) = \frac{E(Y^2) - 16}{4} = \frac{V(Y) + (EY)^2 - 16}{4} = \frac{16 + 16^2 - 16}{4} = \frac{16^2}{4} = 4 \cdot 16 = 64.$$

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Problem 3. (24 points.) Each of two coins, one with P("*Heads*") = 0.6 and the other with P("*Heads*") = 0.002 is tossed 500 times. Assume the result of any coin flip to be independent of any other coin flip. Let X_1 be the number of times the first coin shows heads. Let X_2 be the number of times the second coin shows heads.

(a) What's the distribution of X_1? What about X_2? (Give a common distribution name and its parameters, or write the p.m.f.)

(b) What's the expected value of X_1? What about X_2?

(c) What's the variance of X_1? What about X_2?

(d) Use appropriately the Poisson or Normal approximation to estimate $\mathrm{P}(X_1 \leq 325, X_2 = 4)$ numerically. You may find the table at the end of the exam useful.

Solution:

(a) (6 points.) $X_1 \sim \text{Binomial}(n = 500, p = 0.6)$, and $X_2 \sim \text{Binomial}(n = 500, p = 0.002)$.

(b) (4 points.) $E(X_1) = 500 \cdot 0.6 = 300$, and $E(X_2) = 500 \cdot 0.002 = 1$.

(c) (4 points.) $V(X_1) = 500 \cdot 0.6 \cdot 0.4 = 120$, and $V(X_2) = 300 \cdot 0.002 \cdot 0.998 = 0.998$.

(d) (10 points.)

Solution I.

$$\begin{aligned}
\mathrm{P}(X_1 \leq 325, X_2 = 4) &= \mathrm{P}(X_1 \leq 325) \cdot \mathrm{P}(X_2 = 4) \\
&= \mathrm{P}\left(\frac{X_1 - 300}{\sqrt{120}} \leq \frac{325 - 300}{\sqrt{120}}\right) \cdot \mathrm{P}(X_2 = 4) \\
&\approx \Phi\left(\frac{25}{\sqrt{120}}\right) \cdot \frac{1^4 \cdot e^{-1}}{4!} \\
&\approx \frac{\Phi(2.28)}{e \cdot 4!} \\
&\approx \frac{0.9887}{e \cdot 4!} \\
&\approx 0.015.
\end{aligned}$$

Solution II.

$$\begin{aligned}
\mathrm{P}(X_1 \leq 325.5, X_2 = 4) &= \mathrm{P}(X_1 \leq 325.5) \cdot \mathrm{P}(X_2 = 4) \\
&= \mathrm{P}\left(\frac{X_1 - 300}{\sqrt{120}} \leq \frac{325.5 - 300}{\sqrt{120}}\right) \cdot \mathrm{P}(X_2 = 4) \\
&\approx \Phi\left(\frac{25.5}{\sqrt{120}}\right) \cdot \frac{1^4 \cdot e^{-1}}{4!} \\
&\approx \frac{\Phi(2.32)}{e \cdot 4!} \\
&\approx \frac{0.9898}{e \cdot 4!} \\
&\approx 0.015.
\end{aligned}$$

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Problem 4. (28 points.) Let X and Y be a random variables with joint p.d.f.:

$$f_{X,Y}(x,y) = \begin{cases} c \cdot (2x+y) & , 0 < x < y < 1; \\ 0 & , \text{otherwise}; \end{cases}$$

for a suitable constant c.

(a) Find the constant c.
(b) Find the marginal p.d.f. of X.
(c) Find $\mathrm{E}[Y]$.
(d) Are X and Y independent? Justify your answer.

Solution:

(a) (7 points.) c must be so that

$$\begin{aligned} 1 &= \int_0^1 \int_0^y c\,(2x+y)\,dx\,dy \\ &= c\int_0^1 \left(x^2+xy\right)\Big|_{x=0}^{x=y} dy \\ &= c\int_0^1 2y^2\,dy \\ &= c\left(\frac{2y^3}{3}\Big|_{y=0}^{y=1}\right) \\ &= \frac{2c}{3}. \end{aligned}$$

Hence, $c = 3/2$.

(b) (7 points.)

$$\begin{aligned} f_X(x) &= \int_x^1 \frac{3}{2}\,(2x+y)\,dy\,, \qquad \text{for } 0 < x < 1 \\ &= \int_x^1 3x + \frac{3y}{2}\,dy \\ &= \left(3xy + \frac{3y^2}{4}\right)\Big|_{y=x}^{y=1} \\ &= 3x + \frac{3}{4} - \left(3x^2 + \frac{3x^2}{4}\right) \\ &= 3x + \frac{3}{4} - \frac{15x^2}{4} \end{aligned}$$

Summarizing:

$$f_X(x) = \begin{cases} \frac{-15x^2}{4} + 3x + \frac{3}{4}, & 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(c) (7 points.)

Solution I. Observe that

$$f_Y(y) = \int_0^y 3x + \frac{3y}{2}\,dx = \left(\frac{3x^2}{2} + \frac{3xy}{2}\right)\Big|_{x=0}^{x=y} = 3y^2, \text{ for } 0 < y < 1.$$

So:

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_0^1 y\left(3y^2\right)\, dy = \int_0^1 3y^3\, dy = \frac{3y^4}{4}\Big|_{y=0}^{y=1} = \frac{3}{4}.$$

Solution II.

$$\begin{aligned}
\mathrm{E}[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy \\
&= \int_0^1 \int_0^y y\left(3x + \frac{3y}{2}\right)\, dx\, dy \\
&= \int_0^1 y\left(\frac{3x^2}{2} + \frac{3xy}{2}\Big|_{x=0}^{x=y}\right)\, dy \\
&= \int_0^1 y\left(3y^2\right)\, dy \\
&= \int_0^1 3y^3\, dy \\
&= \frac{3y^4}{4}\Big|_{y=0}^{y=1} = \frac{3}{4}.
\end{aligned}$$

Solution III.

$$\begin{aligned}
\mathrm{E}[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dy\, dx \\
&= \int_0^1 \int_x^1 y\left(3x + \frac{3y}{2}\right)\, dy\, dx \\
&= \int_0^1 \int_x^1 3xy + \frac{3y^2}{2}\, dy\, dx \\
&= \int_0^1 \left(\frac{3xy^2}{2} + \frac{y^3}{2}\Big|_{y=x}^{y=1}\right)\, dx \\
&= \int_0^1 \frac{3x}{2} + \frac{1}{2} - 2x^3\, dx \\
&= \left(\frac{3x^2}{4} + \frac{x}{2} - \frac{x^4}{2}\right)\Big|_{x=0}^{x=1} = \frac{3}{4}.
\end{aligned}$$

(d) (7 points.)

Solution I. No, they are not independent. The joint p.d.f. cannot be factored into $g(x)h(y)$ which is sufficient to imply dependence.

Solution II. No because (X, Y) can only take values such that $0 \leq X \leq Y \leq 1$; in particular, X restricts Y and vice-versa.

Solution III. No because (X, Y) may be any point in the triangle with vertices $(0, 0)$, $(1, 1)$ and $(0, 1)$, in particular, if X takes a value near 1 then Y must also take a value near 1 even though X and Y may take any value between 0 and 1.

(Use the back page if additional space is needed!)

Bonus Problem. (Recover up to 4 points marked down in problems 1-4.) Let X and Y be independent random variables, each uniformly distributed on the interval $(0, 1)$. Find the probability that $|X - Y| \le 0.25$.

Solution:

Solution I. Since (X, Y) is uniformly distributed on the square in the xy-plane with coordinates $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, which has area 1, the probability that (X, Y) belongs to a region in the square is given by its area. Using this geometric argument:

$$\begin{aligned}
\mathrm{P}(|X - Y| \le 0.25) &= 1 - \mathrm{P}(|X - Y| > 0.25) \\
&= 1 - \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right) \times 2 \\
&= 1 - \frac{9}{16} = \frac{7}{16}.
\end{aligned}$$

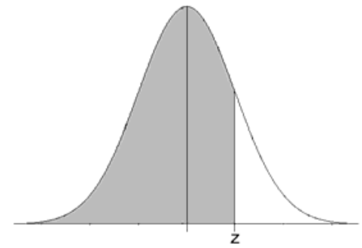
Solution II.

$$\begin{aligned}
\mathrm{P}(|X - Y| \le .25) &= 1 - \mathrm{P}(|X - Y| > 0.25) \\
&= 1 - 2\,\mathrm{P}(Y - X > 0.25) \\
&= 1 - 2\int_0^{.75}\int_{x+.25}^{1} 1\,dy\,dx \\
&= 1 - 2\int_0^{.75} .75 - x\,dx \\
&= 1 - 2\left(\frac{3x}{4} - \frac{x^2}{2}\right)\Big|_0^{.75} \\
&= 1 - 2\left(\left(\frac{3}{4}\right)^2 - \frac{1}{2}\left(\frac{3}{4}\right)^2\right) \\
&= 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}.
\end{aligned}$$

Solution III.

$$\begin{aligned}
\mathrm{P}(|X - Y| \le .25) &= \int_0^{.25}\int_0^{x+.25} 1\,dy\,dx + \int_{.25}^{.75}\int_{x-.25}^{x+.25} 1\,dy\,dx + \int_{.75}^{1}\int_{x-.25}^{1} 1\,dy\,dx \\
&= \int_0^{.25} x + \frac{1}{4}\,dx + \int_{.25}^{.75} \frac{1}{2}\,dx + \int_{.75}^{1} \frac{5}{4} - x\,dx \\
&= \left(\frac{x^2}{2} + \frac{x}{4}\right)\Big|_0^{.25} + \frac{x}{2}\Big|_{.25}^{.75} + \left(\frac{5x}{4} - \frac{x^2}{2}\right)\Big|_{.75}^{1} \\
&= \frac{1}{32} + \frac{1}{16} + \frac{1}{4} + \frac{5}{4} - \frac{1}{2} - \frac{15}{16} + \frac{9}{32} \\
&= \frac{10}{32} - \frac{14}{16} + \frac{6}{4} - \frac{1}{2} \\
&= \frac{5}{16} - \frac{14}{16} + \frac{24}{16} - \frac{8}{16} \\
&= \frac{7}{16}
\end{aligned}$$

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998