APPM 3570/STAT 3100 Exam 1 Spring 2023

NAME: ______________________________

SECTION: 001 *or* 002

Instructions:

1. Calculators are permitted.
2. Notes, your text and other books, cell phones, and other electronic devices are not permitted—except for calculators or as needed to view and upload your work.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5. Don't forget to scan any back pages you used for extra space!
6. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
7. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________ Date: ____________

Duration: 90 minutes

Problem 1. (20 points.) The following five questions pertain to permutations of the following letters: $x, y, y, z, z, z, v, v, v, v, w, w, w, w, w$. Do <u>not</u> simplify your answers.

(a) How many different permutations are there?

(b) How many permutations start with a w and end with an x?

(c) How many permutations keep identical letters together?

(d) How many permutations contain the sub-sequence v, z, v, z, v, z, v?

(e) How many permutations keep no two y's together?

Solution:

(a) (4 points.) $\binom{15}{1,2,3,4,5} = \frac{15!}{1!\cdot 2!\cdot 3!\cdot 4!\cdot 5!}$.

(b) (4 points.) $\binom{13}{2,3,4,4} = \frac{13!}{2!\cdot 3!\cdot 4!\cdot 4!}$.

(c) (4 points.) $5!$.

(d) (4 points.) $\binom{9}{1,2,1,5} = \frac{9!}{1!\cdot 2!\cdot 1!\cdot 5!}$.

(e) (4 points.) $\binom{15}{1,2,3,4,5} - \binom{14}{1,1,3,4,5}$.

(Use the back page if additional space is needed!)

Problem 2. (24 points.) There are three unrelated parts to this question.

(a) Four events occur with probabilities $\mathrm{P}(E) = 0.35$, $\mathrm{P}(F) = 0.15$, $\mathrm{P}(G) = 0.40$, $\mathrm{P}(B) = 0.30$. If $\mathrm{P}(E|B) = 0.20$, what's the probability that exactly one of E or B occurs? Simplify your answer!

(b) A drawer has 8 forks, 8 knives, 4 spoons and one spatula. If I draw 10 objects randomly, what's the probability that I get at least one fork and one spatula? Do not simplify your answer.

(c) A fast-food Mexican restaurant sells burritos with a choice of up to seven different fillings. If customers are equally likely to ask for any combination of at least one and all the possible fillings, what's the probability a new customer asks for a burrito with all seven fillings? Simplify your answer!

Solution:

(a) (8 points.)

$$\begin{aligned}\mathrm{P}(BE^c \cup EB^c) &= \mathrm{P}(BE^c) + \mathrm{P}(EB^c)\\ &= \mathrm{P}(B) - \mathrm{P}(EB) + \mathrm{P}(E) - \mathrm{P}(EB)\\ &= 0.65 - 2 \cdot \mathrm{P}(EB).\end{aligned}$$

But $\mathrm{P}(EB) = \mathrm{P}(E|B) \cdot \mathrm{P}(B) = (0.2)(0.3) = 0.06$. So $\mathrm{P}(BE^c \cup EB^c) = 0.65 - 0.12 = 0.53$.

(b) (8 points.)

$$\begin{aligned}&\mathrm{P}(\text{at least one fork and one spatula are selected})\\ &\quad= 1 - \big[\mathrm{P}(\text{no forks are selected})\\ &\qquad\qquad + \mathrm{P}(\text{no spatulas are selected}) - \mathrm{P}(\text{neither a fork nor a spatula is selected})\big]\\ &\quad= 1 - \left[\frac{\binom{13}{10}}{\binom{21}{10}} + \frac{\binom{20}{10}}{\binom{21}{10}} - \frac{\binom{12}{10}}{\binom{21}{10}}\right].\end{aligned}$$

(c) (8 points.)

$$\frac{1}{\sum_{k=1}^{7}\binom{7}{k}} = \frac{1}{\sum_{k=0}^{7}\binom{7}{k} - 1} = \frac{1}{\sum_{k=0}^{7}\binom{7}{k}1^k 1^{7-k} - 1} = \frac{1}{(1+1)^7 - 1} = \frac{1}{2^7 - 1} = \frac{1}{127}.$$

(Alternatively, $\sum_{k=1}^{7}\binom{7}{k} = 7 + 21 + 35 + 35 + 21 + 7 + 1 = 127$.)

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Problem 3. (24 points.) A manufacturer produces vehicle batteries, some of which are defective. Assume that the probability that a battery is defective is 0.10. There is an electronic test to determine if a battery is or not defective. When the electronic test is conducted on a defective battery, the probability that the electronic test will be positive (i.e., indicate that the battery is defective) is 0.9. Instead, when the electronic test is conducted on a non-defective battery, the probability that the electronic test will be positive is 0.1.

(a) If the electronic test is conducted on a randomly selected battery, what is the probability that the test is positive? Simplify your answer!

(b) Given that the test is positive, what is the probability that the battery is defective? Simplify your answer!

(c) A second test is available. It is a digital test whose results are independent of the electronic test results. If a battery is defective, the second test will find it defective with probability 0.8. Instead, if a battery is non-defective, the digital test will find it defective with probability 0.2. Given that a battery tests positive on both tests, what is the probability that it is defective? Simplify your answer!

Solution:

(a) (8 points.) Consider the events

$$T := \text{"the electronic test is positive"}$$
$$D := \text{"the battery is defective."}$$

Then $\mathrm{P}(T) = \mathrm{P}(T|D)\mathrm{P}(D) + \mathrm{P}(T|D^c)\mathrm{P}(D^c) = (0.9)(0.1) + (0.1)(0.9) = 0.18.$

(b) (8 points.) $\mathrm{P}(D|T) = \frac{\mathrm{P}(DT)}{\mathrm{P}(T)} = \frac{\mathrm{P}(T|D)\mathrm{P}(D)}{\mathrm{P}(T)} = \frac{(0.9)(0.1)}{0.18} = \frac{0.09}{0.18} = 0.5.$

(c) (8 points) Let T_2 be the event that "the digital test is positive." Then

$$\mathrm{P}(D|TT_2) = \frac{\mathrm{P}(DTT_2)}{\mathrm{P}(TT_2)} = \frac{\mathrm{P}(TT_2|D)\mathrm{P}(D)}{\mathrm{P}(TT_2|D)\mathrm{P}(D) + \mathrm{P}(TT_2|D^c)\mathrm{P}(D^c)}$$
$$= \frac{(.9)(.8)(.1)}{(.9)(.8)(.1) + (.1)(.2)(.9)} = \frac{0.072}{0.09} = 0.8.$$

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Problem 4. (32 points.) A biased coin is twice more likely to come up heads than tails. Let G be the number of heads minus the number of tails observed when the coin is tossed independently three times.

(a) What's the probability of flipping heads in one coin toss?

(b) Determine the probability mass function (p.m.f.) of G.

(c) Find $\mathrm{P}(-1 \leq G \leq 2)$.

(d) Determine the cumulative distribution function (c.d.f.) of G.

Solution:

(a) (3 points.) $P(\text{"Heads"}) = 2 \cdot P(\text{"Tails"})$ but $P(\text{"Heads"}) + P(\text{"Tails"}) = 1$. So $P(\text{"Heads"}) = 2/3$.

(b) (12 points.) Observe that $G = H - (3 - H) = 2H - 3$, where H is the number of heads in the three flips. In particular, since $H \in \{0, 1, 2, 3\}$, $G \in \{-3, -1, 1, 3\}$. Futher

$$P(G = -3) = P(H = 0) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27};$$
$$P(G = -1) = P(H = 1) = 3 \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9};$$
$$P(G = 1) = P(H = 2) = 3 \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9};$$
$$P(G = 3) = P(H = 3) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}.$$

The p.m.f. of G is

$$P(G = g) = \begin{cases} 1/27, & \text{if } g = -3; \\ 2/9, & \text{if } g = -1; \\ 4/9, & \text{if } g = 1; \\ 8/27, & \text{if } g = 3; \\ 0, & \text{otherwise.} \end{cases}$$

(c) (5 points.)

$$\mathrm{P}(-1 \leq G \leq 2) = \mathrm{P}(G = -1 \text{ or } G = 1) = \mathrm{P}(G = -1) + \mathrm{P}(G = 1) = 2/9 + 4/9 = 2/3.$$

(d) (12 points.) From part (b), we find that the c.d.f. of G is

$$F_G(g) = \mathrm{P}(G \leq g) = \begin{cases} 0, & \text{for } g < -3; \\ 1/27, & \text{for } -3 \leq g < -1; \\ 7/27, & \text{for } -1 \leq g < 1; \\ 19/27, & \text{for } 1 \leq g < 3; \\ 1, & \text{for } g \geq 3. \end{cases}$$

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Bonus Problem. (Recover up to 4 points marked down in problems 1-4.) Let A, B, and C be independent events. Are $(A \cup B)$ and C independent? Justify your answer with a mathematical argument or a counter-example.

Solution:

Solution I. Since A, B, C are independent, $P(A|C) = P(A)$, $P(B|C) = P(B)$, and $P(AB|C) = P(AB)$. Hence, using the conditional version of the inclusion-exclusion formula:

$$\begin{aligned} P(A \cup B|C) &= P(A|C) + P(B|C) - P(AB|C) \\ &= P(A) + P(B) - P(AB) = P(A \cup B), \end{aligned}$$

so $A \cup B$ and C are independent.

Solution II. $A \cup B$ and C are independent because

$$\begin{aligned} P((A \cup B) \cap C) &= P(AC \cup BC) \\ &= P(AC) + P(BC) - P(AC \cap BC) \\ &= P(A) \cdot P(C) + P(B) \cdot P(C) - P(ABC) \\ &= P(A) \cdot P(C) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C) \\ &= \big(P(A) + P(B) - P(A) \cdot P(B)\big) \cdot P(C) \\ &= \big(P(A) + P(B) - P(AB)\big) \cdot P(C) \\ &= P(A \cup B) \cdot P(C). \end{aligned}$$