Final Exam APPM 3570/STAT 3100 Section 001 Spring 2021

Instructions:

1. Notes, your text and other books, cell phones, and other electronic devices are not permitted, except as needed to view and upload your work, and except calculators.
2. Calculators are permitted.
3. Justify your answers, show all work.
4. Simplify your answers.
5. Start each problem on a new page.
6. When you have completed the exam, send a message through chat to your proctor. Your proctor will then give you the okay to scan your exam and upload it to Gradescope. Verify that everything has been uploaded correctly and the pages have been associated to the correct problem before you leave the zoom proctoring room.
7. Write your full name and section number on your page one.
8. Under your name on page one, write the honor code statement given in the box below and sign and date it.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.
Signature: ______________________ Date: ____________

Problems:

1. (40 points out of 140 total points) There are 3 unrelated parts to this question.
   (a) Suppose that 3 marbles are chosen without replacement from an urn containing 2 white and 6 red marbles. Let X_i equal 1 if the ith marble selected is white and let it equal 0 otherwise. Give the joint pmf of $(X_1,\ X_3)$. Simplify your answers.
   (b) The number of calories in a burrito on the lunch menu is approximately normally distributed with a mean of 300 and a standard deviation of 6.
      i. What is the probability that a randomly chosen burrito will contain more than 305 calories?
      ii. Sam orders 10 burritos for a party. Assuming independence, find the probability that the total calories in the 10 burritos will exceed 3,050.
      iii. If the 10 burritos are served one at a time to 10 guests, what is the probability that the first guest to be served a burrito with over 305 calories is the ninth guest? Explain.

(c) Suppose X is a continuous random variable with pdf

$$f_X(x) = \begin{cases} \frac{3x}{8} + \frac{3x^2}{16} & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

and let $g(x) = x^2$. Find the pdf of the random variable $Y = g(X)$.

Solution:

(a) (13 points) The possible outcomes of X_1 and X_3 are:

Outcome	X_1	X_3	Prob.
WWR	1	0	2/8 * 1/7 * 6/6 = 12/336
WRW	1	1	2/8 * 6/7 * 1/6 = 12/336
WRR	1	0	2/8 * 6/7 * 5/6 = 60/336
RWW	0	1	6/8 * 2/7 * 1/6 = 12/336
RWR	0	0	6/8 * 2/7 * 5/6 = 60/336
RRW	0	1	6/8 * 5/7 * 2/6 = 60/336
RRR	0	0	6/8 * 5/7 * 4/6 = 120/336

The joint pmf of X_1 and X_3 is:

		X_3	
		0	1
X_1	0	180/336	72/336
	1	72/336	12/336

The joint pmf of X_1 and X_3 after simplification is:

		X_3	
		0	1
X_1	0	15/28	6/28
	1	6/28	1/28

The joint pmf of X_1 and X_3 in decimal form is:

		X_3	
		0	1
X_1	0	.5357	.2143
	1	.2143	.0357

(b) (15 points)

i. (5 points) We are given that $X_i \sim \mathcal{N}(300, \sigma^2 = 36)$, for $i = 1, 2, \ldots$. Therefore, $\text{P}(X_i > 305) = \text{P}(Z > \frac{305-300}{6}) = \text{P}(Z > .8333) \approx 1 - \Phi(.83) = 1 - .7967 = .2033$.

ii. (5 points) Let $Y = \sum_1^{10} X_i$ be the total calories in ten servings of burritos. Then, assuming independence of X_i, we have $Y \sim \mathcal{N}(3000, \sigma^2 = 360)$ and $\text{P}(Y > 3050) = \text{P}(Z > \frac{3050-3000}{18.97367}) = \text{P}(Z > 2.635) \approx 1 - \Phi(2.64) = 1 - .9959 = .0041$.

iii. (5 points) Let S be the number of servings required until the first time a guest is served a burrito with over 305 calories, then $S \sim \text{Geom}(.2033)$. $\text{P}(S = 9) = (1 - .2033)^8(.2033) \approx .033$.

(c) (12 points) Since X is a continuous random variable, and $g(x)$ is a strictly monotonic differentiable function of x. Then

$$f_Y(y) = f_X(x)[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right| \quad \text{for } y = g(x)$$

$$g^{-1}(y) = \sqrt{y}$$

$$\left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \begin{cases} \frac{3}{16} + \frac{3\sqrt{y}}{32} & 1 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

2. (40 points) Let (X, Y) be jointly distributed random variables with conditional pdf given by:

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{x^2} & 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

and marginal pdf of X given by:

$$f_X(x) = \begin{cases} 5x^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint pdf of X and Y.

(b) Find the marginal pdf of Y.

(c) Find $E[Y|X]$ and use it to find the expectation of Y.

(d) Find $\text{Cov}(X, Y)$.

Solution:

(a) (10 points)

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x)$$

$$f_{X,Y}(x, y) = \begin{cases} 10x^2 y & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (10 points)

$$\begin{aligned} f_Y(y) &= \int_y^1 10x^2 y \; dx \\ &= \frac{10x^3 y}{3} \Big|_y^1 \\ &= \frac{10y}{3} - \frac{10y^4}{3} \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{10y}{3} - \frac{10y^4}{3} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) (10 points)

$$\begin{aligned} E[Y|X] &= \int_0^x \frac{2y^2}{x^2} \ dy \\ &= \frac{2y^3}{3x^2} \Big|_0^x \\ &= \frac{2x}{3} \end{aligned}$$

$$\begin{aligned} E[Y] = E[E[Y|X]] &= \int_0^1 \frac{2x}{3} 5x^4 \ dx \\ &= \frac{10x^6}{18} \Big|_0^1 \\ &= \frac{10}{18} = \frac{5}{9} \end{aligned}$$

(d) (10 points)

$$\begin{aligned} E[XY] &= \int_0^1 \int_y^1 10x^3y^2 \ dx \, dy \\ &= \int_0^1 \frac{10x^4y^2}{4} \Big|_y^1 \ dy \\ &= \int_0^1 \frac{10y^2}{4} - \frac{10y^6}{4} \ dy \\ &= \frac{5y^3}{6} - \frac{5y^7}{14} \Big|_0^1 \\ &= \frac{5}{6} - \frac{5}{14} = \frac{35}{42} - \frac{15}{42} = \frac{20}{42} \\ &= \frac{10}{21} \end{aligned}$$

$$\begin{aligned} E[X] &= \int_0^1 5x^5 \ dx \\ &= \frac{5x^6}{6} \Big|_0^1 \\ &= \frac{5}{6} \end{aligned}$$

$$\begin{aligned} \text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\ &= \frac{10}{21} - \left(\frac{5}{6}\right)\left(\frac{5}{9}\right) \\ &= \frac{10}{21} - \frac{25}{54} \\ &= \frac{5}{378} \approx .0132 \end{aligned}$$

3. (25 points) If X and Y are independent and identically distributed uniform random variables on (0, 1), compute the joint density of $U = X + Y,\ \ V = X/(X + Y)$.

Solution:

(8 points)

$$f_{X,Y}(x, y) = \begin{cases} 1 & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} X &= VU \\ Y &= U - UV \end{aligned}$$

(8 points)

$$\text{J} = \begin{vmatrix} v & 1 - v \\ u & -u \end{vmatrix}$$

$$\begin{aligned} &= -u \\ |\text{J}| &= | - u| = u \end{aligned}$$

(9 points)

$$f_{U,V}(u, v) = \begin{cases} u & 0 < vu, u - vu < 1 \\ 0 & \text{otherwise} \end{cases}$$

For $0 < u < 1,$ then $0 < v < 1$
For $1 < u < 2,$ then $1 - 1/u < v < 1/u$

4. (35 points) Recall that if $X \sim \text{Exp}(\lambda)$, then $E[X] = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$.
Recall that if $Y \sim \text{Unif}(0, 1)$, then $E[Y] = 1/2$ and $\text{Var}(Y) = 1/12$.
Mike has 200 batteries whose lifetimes are independent exponential random variables, each with a mean of 4 hours.

(a) If the batteries are used one at a time, with a failed battery being replaced immediately by a new one, approximate the probability that there is still a working battery after 810 hours.

(b) Suppose that the time it takes (in hours) to replace a failed battery is uniformly distributed over (0, .8), independently. Approximate the probability that all batteries have failed before 1000 hours have passed.

Solution:

(a) (15 points)

Let X_i be the lifetime of the ith battery, $i = 1, 2, \ldots, 200$.
$E[X_i] = 4$, $\text{Var}(X_i) = 16$
By the CLT, $\sum_{n=1}^{200} X_i \overset{approx}{\sim} \mathcal{N}(800, 3200)$

$$\begin{aligned}
\text{P}(\sum_{n=1}^{200} X_i > 810) &\approx \text{P}\Big(Z > \frac{810 - 800}{\sqrt{3200}}\Big) \\
&= \text{P}(Z > .1768) \\
&\approx 1 - \Phi(.18) \\
&= 1 - .5714 = .4286
\end{aligned}$$

(b) (20 points)

Let R_i be the time needed to replace the ith battery, $i = 1, 2, \ldots, 200$.
$E[R_i] = .4$, $\text{Var}(R_i) = .8^2 * 1/12 = .0533$

$E[\sum_{n=1}^{199} R_i] = 79.6$, $\text{Var}(\sum_{n=1}^{199} R_i) = 10.6067$

By the CLT, $\sum_{n=1}^{200} X_i + \sum_{n=1}^{199} R_i \overset{approx}{\sim} \mathcal{N}(879.6,\ 3210.61)$

$$\begin{aligned}
\text{P}(\sum_{n=1}^{200} X_i + \sum_{n=1}^{199} R_i < 1000) &\approx \text{P}\Big(Z < \frac{1000 - 879.6}{\sqrt{3210.61}}\Big) \\
&= \text{P}(Z < 2.1249) \\
&\approx \Phi(2.12) \\
&= .983
\end{aligned}$$

THE END