# APPM 3570/STAT 3100 Exam 2 Spring 2021

**Instructions:**

1. Notes, your text and other books, cell phones, and other electronic devices are not permitted, except as needed to view and upload your work, and except calculators.
2. Calculators are permitted.
3. Justify your answers, show all work.
4. Simplify your answers.
5. Start each problem on a new page.
6. When you have completed the exam, send a message through chat to your proctor. Your proctor will then give you the okay to scan your exam and upload it to Gradescope. Verify that everything has been uploaded correctly and the pages have been associated to the correct problem before you leave the zoom proctoring room.
7. Write your full name and section number on your page one.
8. Under your name on page one, write the honor code statement given in the box below and sign and date it.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________ Date: ____________

**Problems:**

1. (35 points) There are 4 unrelated parts to this question.

   (a) An ice chest contains 12 beverages covered by ice. Three of the beverages are bottled water, four are bottled tea and there are five bottles of root beer. Two beverages are chosen at random. Let $W$ be the number of bottles of water that are chosen and let $T$ be the number of bottles of tea that are chosen. Find the the joint probability mass function (pmf) of $W$ and $T$.

   (b) Suppose you play a board game and you land in "jail" and the only way to get out of "jail" is to roll doubles using two fair six-sided dice (regular dice).

      i. What is the probability that it takes exactly 3 attempts (3 rolls of the two dice) to get out of "jail"?

      ii. What is the expected number of attempts needed to get out of "jail"?

      iii. What is the probability that it takes more than the expected number of attempts before you get out of "jail"?

   (c) The time in minutes for a laser to detect a particle is an exponentially distributed random variable. If on average 8 particles are detected every 17 minutes, what is the conditional probability that the length of time until a particle is detected is greater than 3 minutes given that the length of time exceeds 1 minute?

   (d) Let $X$ be a random variable and $\mathrm{P}(X = 1) = 1 - \mathrm{P}(X = 0)$. If $5\mathrm{Var}(X) = E[X]$, find $\mathrm{P}(X = 0)$.

**Solution:**

(a) (8 points) The joint of $W$ and $T$ is

$$P(W=0,\, T=0) = \binom{5}{2}/\binom{12}{2} = 10/66, \quad P(W=1,\, T=0) = \binom{3}{1}\binom{5}{1}/\binom{12}{2} = 15/66,$$
$$P(W=2,\, T=0) = \binom{3}{2}/\binom{12}{2} = 3/66, \quad P(W=0,\, T=1) = \binom{4}{1}\binom{5}{1}/\binom{12}{2} = 20/66,$$
$$P(W=1,\, T=1) = \binom{3}{1}\binom{4}{1}/\binom{12}{2} = 12/66, \quad P(W=0,\, T=2) = \binom{4}{2}/\binom{12}{2} = 6/66$$

(b) (11 points)

i. The probability of rolling doubles on one roll of two fair dice is $1/6$. If $X$ is the number of rolls until a double appears, then $X \sim \text{Geom}(p = 1/6)$. Then $P(X = 3) = (5/6)^2(1/6) \approx .1157$.

ii. $E[X] = 1/p = 6$ attempts

iii. $P(X > 6) = \frac{(5/6)^6(1/6)}{1-5/6} = (5/6)^6 \approx .3349$

(c) (8 points) $P(X > 3|X > 1) = P(X > 2) =$

$$1 - \int_{-\infty}^{2} f(x)dx = 1 - \int_{0}^{2} \frac{8}{17} e^{-\frac{8x}{17}}\, dx = 1 - \Big[ -e^{-\frac{8x}{17}} \Big|_0^2 \Big] = 1 - (-e^{-\frac{16}{17}} + 1)$$
$$= e^{-\frac{16}{17}} \approx .39017$$

(d) (8 points) Let $p = P(X = 1)$. $E[X] = p$. $E[X^2] = p$ Therefore $\text{Var}(X) = p - p^2$. If $5\text{Var}(X) = E[X]$, then $5(p - p^2) = p$ which implies $p = 4/5$ and $P(X = 0) = 1/5$.

2. (30 points) A cross country skier named Alex regularly skis a 15 mile route along the Owl Creek trail, and always hears the laughing call of the Pinyon Jay somewhere along the route at a random location uniformly distributed over the 15 miles. Let $J$ be the location along the route at which Alex hears the bird call. $J \sim \mathcal{U}(0, 15)$.

(a) Write the probability density function (pdf) of $J$.

(b) Lately, due to recent snowfalls, Alex now hears the bird at a location along the route given by $Y = g(J)$ where

$$g(j) = 15\Big(1 - \frac{j}{15}\Big)^{1/2}$$

Find the pdf of the new location where Alex hears the Jay. Be sure to define the pdf for all values on the real line.

(c) Under the situation in part (b), find the probability that Alex hears the Jay with less than 4 miles left on the route.

(d) Under the situation in part (b), at what location along the route does Alex expect to hear the bird call?

**Solution:**

(a) (7 points)

$$f_J(j) = \begin{cases} \frac{1}{15} & 0 < j < 15 \\ 0 & \text{otherwise} \end{cases}$$

(b) (9 points) Using the method where we begin with the CDF of $Y$,

$$\begin{aligned} F_Y(a) &= \mathrm{P}(Y \leq a) = \mathrm{P}\Big(15\Big(1 - \frac{J}{15}\Big)^{1/2} \leq a\Big) \\ &= \mathrm{P}\Big(J \geq 15 - \frac{a^2}{15}\Big) \quad 0 < a < 15 \\ &= 1 - \int_0^{15-\frac{a^2}{15}} f(j)dj \end{aligned}$$

$$\begin{aligned} f_Y(a) &= \frac{d}{da}(F_Y(a)) \\ &= -\Big(-\frac{2a}{15}\Big)\frac{1}{15} \\ &= \frac{2a}{225} \quad 0 < a < 15 \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{2y}{225} & 0 < y < 15 \\ 0 & \text{otherwise} \end{cases}$$

Note that when $j = 0$, then $15\Big(1 - \frac{j}{15}\Big)^{1/2} = 15$ And when $j = 15$, then $15\Big(1 - \frac{j}{15}\Big)^{1/2} = 0$. Also note that $d/dj(15\Big(1 - \frac{j}{15}\Big)^{1/2} = -1/2(1 - j/15)^{-1/2} < 0$ so the function is strictly decreasing and $y$ ranges from 0 to 15.

(c) (7 points)

$$\begin{aligned} \mathrm{P}(Y > 11) &= 1 - \int_0^{11} \frac{2y}{225}dy \\ &= 1 - \Big[\frac{2y^2}{450}\Big|_0^{11}\Big] = 1 - \frac{242}{450} \\ &= \frac{104}{225} \approx .4622 \end{aligned}$$

(d) (7 points)

$$\begin{aligned} E[Y] &= \int_0^{15} \frac{2y^2}{225}dy \\ &= \frac{2y^3}{675}\Big|_0^{15} = \frac{2 * 15^3}{15^2 * 3} \\ &= \frac{30}{3} = 10 \text{ miles} \end{aligned}$$

Alex expects to hear the bird call after traveling 10 miles.

3. (35 points) Suppose $X$ and $Y$ are jointly continuous random variables with joint density function given by

$$f_{X,Y}(x, y) = \begin{cases} c\,(20 - x - y) & 0 \leq x \leq 1, \quad 0 \leq y \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of $c$.

(b) Find an expression for $\text{P}(Y > X^2)$. In other words write the expression as an integral of a function of one or more variables, BUT DO NOT EVALUATE.

(c) Find $f_Y$, the marginal density function for $Y$.

(d) Set up, but do not compute, the integrals needed to find the variance of $Y$.

(e) Are $X$ and $Y$ independent random variables? Justify or explain your answer.

**Solution:**

(a) (7 points)

$$
\begin{aligned}
1 &= c\int_0^5 \int_0^1 (20 - x - y)dx\, dy = c\int_0^5 (20x - x^2/2 - xy)\Big|_0^1 dy \\
&= c\int_0^5 (39/2 - y)\, dy = c(39y/2 - y^2/2)\Big|_0^5 \\
&= c(195/2 - 25/2) = c(85)
\end{aligned}
$$

Therefore, $c = 1/85$

(b) (8 points)

$$\text{P}(Y > X^2) = \frac{1}{85}\int_0^1 \int_{x^2}^5 (20 - x - y)\, dy\, dx$$

(c) (8 points)

$$
\begin{aligned}
f_Y(y) &= \frac{1}{85}\int_0^1 (20 - x - y)\, dx = \frac{1}{85}(20x - x^2/2 - xy)\Big|_0^1 \\
&= \frac{1}{85}(39/2 - y) \\
&= \frac{39}{170} - \frac{y}{85}
\end{aligned}
$$

$$
f_Y(y) = \begin{cases} \frac{39}{170} - \frac{y}{85} & 0 \le y \le 5 \\ 0 & \text{otherwise} \end{cases}
$$

(d) (7 points)

$$\text{Var}(Y) = \int_0^5 y^2\left(\frac{39}{170} - \frac{y}{85}\right) dy - \left(\int_0^5 y\left(\frac{39}{170} - \frac{y}{85}\right) dy\right)^2$$

(e) (5 points) No $X$ and $Y$ are not independent since $f_{X,Y}$ cannot be factored as $f_X(x)f_Y(y)$.

THE END