APPM 3570/STAT 3100 — Exam 1 — Fall 2024

On the front of your bluebook, write (1) **your name**, (2) **Exam 1**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers. Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:______________________________ Date:__________

1. [Exam01] (40pts) Answer all the problems below. Justify your answers.

   (a) (10pts) In how many ways can we distribute eight identical balls into four distinct urns so that the fourth urn has either two or three balls in it? (The other three urns don't all have to have balls in them.)

   (b) (10pts) (*Water management*) In the town of Niwot, each month of the year receives a different level of rain with the ranking, $A, B, C, ..., K, L$, where A is the month with most rain and L is the month with the least rain. For example, $ABCDEFGHIJKL$ is a year in which January receives the most rain, February the second most, and so on. How many distinct rainfall rankings are there?

   (c) (10pts) (Referring to the *Water management* scenario above.) The town of Niwot floods if the three heaviest months of rain (A, B, C) are adjacent to one another (e.g., BCA, ACB, etc.). Assuming each ranking is equally likely, what is the probability the town floods in a given year? Ignore effects from the previous year.

   (d) (10pts) Suppose that events V and W are independent with $P(V \cup W) = \frac{6}{10}$ and $P(V) = \frac{3}{10}$, find $P(W)$.

Solution: (a) (10pts) There are only two possibilities for the fourth urn, namely that it contains either 2 balls or 3 balls and, for example, the number of ways of distributing 2 balls to the fourth urn and the remaining $8 - 2 = 6$ balls to the remaining three urns is the same as the number of non-negative, integer solutions to $x_1 + x_2 + x_3 = 6$ (or the number of permutations of $* * * * * * //$) which is $\frac{8!}{2!6!} = \binom{6+3-1}{3-1}$ and, so, we have a total of

$$\underbrace{\binom{6+3-1}{3-1}}_{\substack{\text{4th urn}\\ \text{has 2 balls}}} + \underbrace{\binom{5+3-1}{3-1}}_{\substack{\text{4th urn}\\ \text{has 3 balls}}} = \binom{8}{2} + \binom{7}{2} = 49 \text{ ways.}$$

(b) (10pts) Let S be the set of all permutations of the 12 letters then $|S| = 12!$.

(c) (10pts) Consider the single superletter "ABC", then there are $12 - 3 + 1 = 10$ objects to permute and note that there are $3!$ ways to permute the letters in the superletter ABC, thus, the total number of possible permutations with ABC adjacent is $3! \cdot 10!$. The probability of a flood is therefore

$$P\left(\substack{\text{Niwot}\\ \text{floods}}\right) = \frac{3! \cdot 10!}{12!} = \frac{6}{12 \cdot 11} = \frac{1}{22} \approx 0.0455.$$

(d) (10pts) Using the Inclusion-Exclusion Principle and the fact that events V and W are independent, we have

$$\begin{aligned} P(V \cup W) &= P(V) + P(W) - P(V \cap W) \\ &= P(V) + P(W) - P(V)P(W) \;\Rightarrow\; P(W) = \frac{P(V \cup W) - P(V)}{1 - P(V)} = \frac{\frac{6}{10} - \frac{3}{10}}{1 - \frac{3}{10}} = \frac{3}{7} \approx 0.4286. \end{aligned}$$

2. [Exam01] (32pts) Drawers A and B contain green and red pencils as follows:

Drawer A: 6 green pencils and 2 red pencils,
Drawer B: 2 green pencils and 10 red pencils.

A drawer is chosen at random and then a pencil is also chosen at random from that drawer. (Let A be the event that Drawer A is selected and let B be the event that Drawer B is selected, let G be the event a green pencil is selected and let R be the event a red pencil is selected.) Do the following problems:

(a) (8pts) Given that Drawer A is selected, what is the probability of choosing a green pencil?

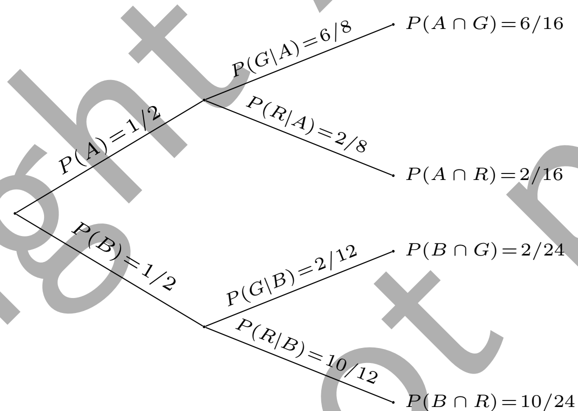
(b) (8pts) Suppose we don't know which drawer is selected, what is the probability that the pencil chosen is green?

(c) (8pts) If the pencil chosen is green, what is the probability that it was chosen from Drawer A?

(d) (8pts) Find $P(A|G^c)$.

Solution:

Probability Tree Diagram of Drawers and Pencils



(a)(8pts) The conditional probability of choosing a green pencil given that Drawer A was selected is $P(G|A) = \dfrac{6}{8}$.

(b)(8pts) We can choose a green pencil in only one of two ways: either from Drawer A or from Drawer B, thus

$$\begin{aligned} P(G) &= P\big((A\cap G)\cup(B\cap G)\big) \\ &= P(A\cap G) + P(B\cap G) \\ &= P(A)P(G|A) + P(B)P(G|B) = \frac{1}{2}\cdot\frac{6}{8} + \frac{1}{2}\cdot\frac{2}{12} = \frac{11}{24} \approx 0.4583. \end{aligned}$$

(c)(8pts) We need to find $P(A|G)$, using Bayes Rule, we have

$$P(A|G) = \frac{P(A\cap G)}{P(G)} = \frac{P(A)P(G|A)}{P(A)P(G|A)+P(B)P(G|B)} = \frac{\frac{1}{2}\cdot\frac{6}{8}}{\frac{11}{24}} = \frac{9}{11} \approx 0.8182.$$

(d)(8pts) Here we have

$$P(A|G^c) = \frac{P(A\cap G^c)}{P(G^c)} = \frac{P(A)P(G^c|A)}{1-P(G)} = \frac{P(A)P(R|A)}{1-P(G)} = \frac{\frac{1}{2}\cdot\frac{2}{8}}{1-\frac{11}{24}} = \frac{3}{13} \approx 0.2308.$$

3. [Exam01] (28pts) Let H be the event that a coin shows *heads* and T if *tails*. A fair coin is tossed independently and repeatedly until the pair HT appears (in this order) for the first time. For example, if the event is accomplished in $k=3$ flips, then this event could happen in two ways: $E = \{HHT, THT\}$.

   (a) (7pts) **Using the set notation established above**, list all the possible ways the event that HT appears (in this order) for the first time can occur for $k=6$ flips.

   (b) (7pts) Let X be the random variable denoting *the total number of coin flips needed for this event to occur*. Clearly $X \in \{2, 3, \ldots\}$ and, for example, $X(THHT) = 4$. Find the probability $P(X = 7)$.

   (c) (7pts) Find the *probability mass function* of X, $p(k) = P(X=k)$. Be sure to define the pmf for all $k \in \mathbb{R}$. *(You do not have to verify the pmf.)*

   (d) (7pts) Find the conditional probability $P(X=7 \,|\, X \geq 3)$.

Solution:

(a) (7pts) If the first occurrence of HT is to be at the end of the sequence of flips and, if the first flip is H, then all the flips but the last must be H also, that is, we have $HHHHHT$. If the first flip is not an H, then the maximum allowable occurrences of T in $k = 6$ flips (other than the T at the end) is 4 where any T, save the final T, must be preceded by the symbol T, so, we have

$$E = \{HHHHHT,\ THHHHT,\ TTHHHT,\ TTTHHT,\ TTTTHT\}$$

and observe that $|E| = 5$.

(b) (7pts) If $k = 7$, then the event can be expressed as

$$E = \{HHHHHHT,\ THHHHHT,\ TTHHHHT,\ TTTHHHT,\ TTTTHHT,\ TTTTTHT\}$$

where $|E| = 6$. Because of independence and the uniform distribution of the coin, each occurrence of the event listed in set E has the same probability of occurring, for example,

$$P(HHHHHHT) = P(H)P(H)P(H)P(H)P(H)P(H)P(T) = \left(\frac{1}{2}\right)^7$$

and, in general, $P(HHHHHHT) = P(THHHHHT) = \cdots = P(TTTTTHT) = \left(\frac{1}{2}\right)^7$, thus,

$$P(X = 7) = P(E) = 6 \cdot \left(\frac{1}{2}\right)^7 = \frac{6}{2^7} \approx 0.0469.$$

(c) (7pts) The event $\{X = k\}$ is the event that k flips were required for the first occurrence of HT to be on the final two flips. If the first flip is H, then there is only one possible sequence, namely $HH\cdots HT$. Otherwise, in k flips, we can have a maximum of $k-2$ occurrences of the symbol T (not counting the final T) where each T cannot be preceded by an H so there are only $k-2$ possibilities for this case, namely

$$\underbrace{THH\cdots HHT}_{\#1},\quad \underbrace{TTH\cdots HHT}_{\#2},\ \cdots,\ \underbrace{TTT\cdots THT}_{\#k-2}$$

so we have a total of $1+(k-2) = k-1$ possibilities. Since the coin is fair, the probability of any event involving k flips (by independence) is $\left(\frac{1}{2}\right)^k$, thus, the pmf is

$$p(k) = \begin{cases} (k-1)\cdot\left(\dfrac{1}{2}\right)^k, & \text{for } k = 2, 3, \ldots, \\ 0, & \text{else,} \end{cases} = \begin{cases} \dfrac{k-1}{2^k}, & \text{for } k = 2, 3, \ldots, \\ 0, & \text{else.} \end{cases}$$

(d) (7pts) Proceeding by definition, we have

$$P(X=7|X\geq 3)=\frac{P(\{X=7\}\cap\{X\geq 3\})}{P(X\geq 3)}=\frac{P(X=7)}{1-P(X<3)}$$
$$=\frac{P(X=7)}{1-P(X=2)}=\frac{\frac{6}{2^7}}{1-\left(\frac{1}{2}\right)^2}=\frac{\frac{3}{2^6}}{\frac{3}{4}}=\frac{1}{2^4}.$$