APPM 3570/STAT 3100 — Exam 3 — Fall 2023

On the front of your bluebook, write (1) **your name**, (2) **Exam 3**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:____________________________ Date:_________

1. (30pts) There are 3 unrelated parts to this question. Justify your answers.

   (a) (10pts) A bookshelf contains 8 distinct books, three are probability books, two are statistics books, and three are physics books. Two books are chosen at random. Let R be the number of probability books that are chosen and let T be the number of statistics books that are chosen. Find the conditional probability $P(T = 0 \mid R = 1)$, simplify your answer.

   (b) (10pts) Eight (distinct) new employees are hired by a company. The company has 10 different divisions and will assign each of the new employees randomly to one of the 10 divisions. (A division can get more than one new employee.) What is the expected number of divisions that receive at least one new employee?

   (c) (10pts) If the random variable X, Y and Z have the means $\mu_X = 2$, $\mu_Y = -3$ and $\mu_Z = 4$, the variances $\sigma_X^2 = 1$, $\sigma_Y^2 = 5$ and $\sigma_Z^2 = 2$ and the covariances $\mathrm{cov}(X, Y) = -2$, $\mathrm{cov}(X, Z) = -1$ and $\mathrm{cov}(Y, Z) = 1$. Find the variance of the random variable $W = 3X - Y + 2Z$.

Solution: (a)(10pts) Note that

$$P(T = 0 \mid R = 1) = \frac{P(R = 1, T = 0)}{P(R = 1)} \quad \text{where} \quad P(R = 1, T = 0) = \frac{\binom{3}{1}\binom{2}{0}\binom{3}{1}}{\binom{8}{2}} = \frac{3 \cdot 3}{\frac{8!}{2!6!}} = \frac{9}{28}$$

and

$$P(R = 1) = P(R = 1, T = 0) + P(R = 1, T = 1) = \frac{\binom{3}{1}\binom{2}{0}\binom{3}{1}}{\binom{8}{2}} + \frac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{\binom{8}{2}} = \frac{9 + 6}{28} = \frac{15}{28}$$

so

$$P(T = 0 \mid R = 1) = \frac{P(R = 1, T = 0)}{P(R = 1)} = \frac{9/28}{15/28} = \frac{9}{15} = \frac{3}{5}.$$

(b)(10pts) Let

$$X_i = \begin{cases} 1, & \text{if at least one new employee is in division } i, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i = 1, 2, \ldots, 10,$$

then $\mathrm{P}(X_i = 1) = 1 - \mathrm{P}(X_i = 0) = 1 - \left(\frac{9}{10}\right)^8 = 1 - (0.9)^8$, now let $X = \sum_{i=1}^{10} X_i$ then

$$E(X) = \sum_{i=1}^{10} E(X_i) = \sum_{i=1}^{10} \mathrm{P}(X_i = 1) = \sum_{i=1}^{10} [1 - (0.9)^8] = 10[1 - (0.9)^8].$$

(c)(10pts) Since $W = 3X - Y + 2Z$ and using the fact that $\sigma_X^2 = 1$, $\sigma_Y^2 = 5$ and $\sigma_Z^2 = 2$ with covariances $\mathrm{cov}(X, Y) = -2$, $\mathrm{cov}(X, Z) = -1$ and $\mathrm{cov}(Y, Z) = 1$, we have

$$\begin{aligned} V(W) &= \mathrm{cov}(W, W) \\ &= \mathrm{cov}(3X - Y + 2Z, 3X - Y + 2Z) \\ &= 3^2 V(X) + (-1)^2 V(Y) + 2^2 V(Z) \\ &\quad + 2(3 \cdot -1)\mathrm{cov}(X, Y) + 2 \cdot (3 \cdot 2)\mathrm{cov}(X, Z) + 2(2 \cdot -1)\mathrm{cov}(Y, Z) \\ &= (9 \cdot 1) + 5 + (4 \cdot 2) + (-6 \cdot -2) + (12 \cdot -1) + (-4 \cdot 1) \\ &= 18. \end{aligned}$$

2. (40pts) A nut company markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is 1 lb, but the weight contribution of each type of nut is random. Let X be the weight of almonds in a selected can and Y the weight of cashews. The joint pdf of (X, Y) is given to be

$$f(x,y) = \begin{cases} \frac{24}{5}xy, & \text{for } 0 < x < 1,\ 0 < y < 1, \text{ and } y > 1 - x, \\ 0, & \text{else.} \end{cases}$$

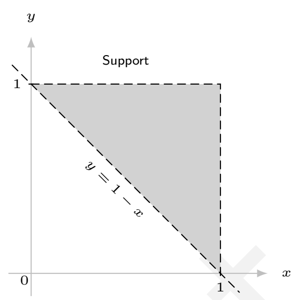
(a) (10pts) Set-up, but do *not* evaluate, a double integral (or integrals) to find $P(Y < X)$.

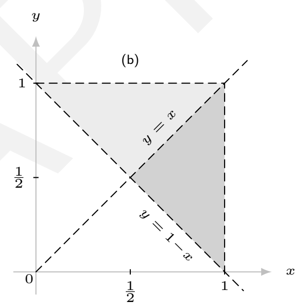
(b) (10pts) Find the marginal pdf $f_X(x)$ and the expectation $E[X]$, be sure to define all pdfs for all values of $\mathbb{R}$.

(c) (10pts) Find the conditional pdf $f_{Y|X}(y|x)$ and the conditional expectation $E[Y|X]$.

(d) (10pts) If we know that $E[75XY] = 38$, find $\text{cov}(X, Y)$.

Solution:





(a)(10pts) Note that $P(Y < X) = P(\frac{1}{2} < X < 1, 1-X < Y < X) = \int_{\frac{1}{2}}^{1}\int_{1-x}^{x} f(x,y)\,dydx$, or, alternately, we could also write

$$\begin{aligned} P(Y < X) &= P(1-Y < X < 1,\ 0 < Y < \tfrac{1}{2}) + P(Y < X < 1,\ \tfrac{1}{2} < Y < 1) \\ &= \int_0^{\frac{1}{2}}\int_{1-y}^{1} f(x,y)\,dxdy + \int_{\frac{1}{2}}^{1}\int_{y}^{1} f(x,y)\,dxdy. \end{aligned}$$

(b)(10pts) For each $0 < x < 1$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{1-x}^{1} \frac{24}{5}xy\,dy = \frac{24}{5}x \cdot \frac{y^2}{2}\bigg|_{1-x}^{1} = \frac{12}{5}x\big[1 - (1-x)^2\big] = \frac{12}{5}x\big(2x - x^2\big) = \frac{12}{5}x^2\big(2 - x\big),$$

thus, the marginal pdf of X is $\boxed{f_X(x) = \frac{12}{5}x^2\big(2 - x\big), \text{ for } x \in (0,1) \text{ and } 0 \text{ otherwise.}}$

The expectation of X is

$$\begin{aligned} E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx &= \int_0^1 x \cdot \frac{12}{5}x^2(2-x)\,dx = \frac{12}{5}\int_0^1 (2x^3 - x^4)\,dx \\ &= \frac{12}{5}\Big(2\frac{x^4}{4} - \frac{x^5}{5}\Big)\bigg|_0^1 = \frac{12}{5}\Big(\frac{2}{4} - \frac{1}{5}\Big) = \frac{12}{5}\Big(\frac{10}{20} - \frac{4}{20}\Big) = \frac{12}{5}\Big(\frac{6}{20}\Big) = \frac{18}{25}. \end{aligned}$$

(c)(10pts) For each $x \in (0,1)$, the conditional pdf of $Y|X$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \dfrac{\frac{24}{5}xy}{\frac{12}{5}x^2(2-x)}, & \text{for } 1-x < y < 1, \\ 0, & \text{else.} \end{cases} = \begin{cases} \dfrac{2y}{x(2-x)}, & \text{for } 1-x < y < 1, \\ 0, & \text{else.} \end{cases}$$

The conditional expectation of $Y|X$, for each $x \in (0,1)$, is

$$
\begin{aligned}
E[Y|X] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy = \int_{1-x}^{1} y \cdot \frac{2y}{x(2-x)}\,dy &= \frac{2}{x(2-x)} \int_{1-x}^{1} y^2\,dy \\
&= \frac{2}{x(2-x)} \cdot \left.\frac{y^3}{3}\right|_{1-x}^{1} \\
&\boxed{= \frac{2}{3x(2-x)}\left[1-(1-x)^3\right]} \\
&= \frac{2}{3x(2-x)}\left[1-(1-3x+3x^2-x^3)\right] \\
&= \frac{2}{3x(2-x)}\left[3x-3x^2+x^3\right] \\
&= \frac{2x(3-3x+x^2)}{3x(2-x)} \boxed{= \frac{2(3-3x+x^2)}{3(2-x)} \text{ for } x \in (0,1).}
\end{aligned}
$$

(d)(10pts) Since $E[75XY] = 38$, we have $75E[XY] = 38 \Rightarrow E[XY] = \frac{38}{75}$, and, by symmetry, we have $E[Y] = E[X] = \frac{18}{25}$, so

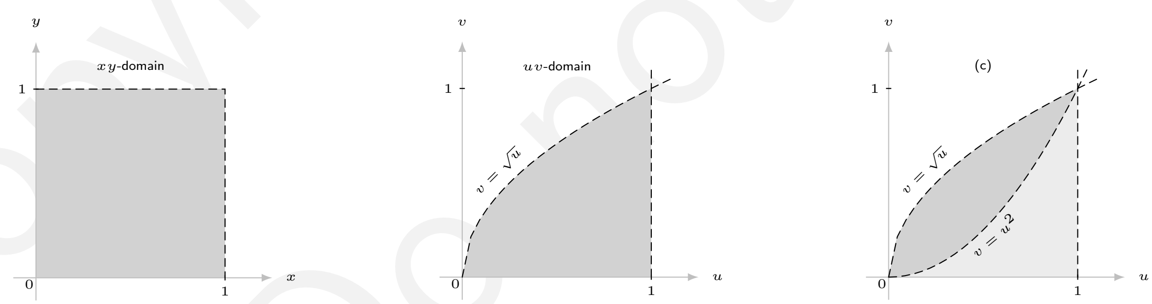
$$
\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{38}{75} - \left(\frac{18}{25}\right)^2.
$$

3. (30pts) If the joint probability density of X and Y is given by

$$
f(x,y) = \begin{cases} 4xy, & \text{for } 0<x<1,\ 0<y<1, \\ 0, & \text{elsewhere.} \end{cases}
$$

   (a) (10pts) Find the joint probability density function of (U,V) where $U = X^2$ and $V = XY$. Specify the domain.
   (b) (10pts) Set-up, but do *not* evaluate, an integral (or integrals) to find the probability $P(U^2 < V)$
   (c) (10pts) Find the marginal probability density function of $U = X^2$.

Solution:



(a)(10pts) First, we solve for X and Y, note that

$$
X > 0 \text{ and } U = X^2 \ \Rightarrow\ X = \sqrt{U} \ \Rightarrow\ \begin{cases} X = \sqrt{U}, \\ Y = \dfrac{V}{X} = \dfrac{V}{\sqrt{U}}. \end{cases}
$$

For the domain note that

$$
0 < x < 1 \ \Rightarrow\ 0 < \sqrt{u} < 1 \ \Rightarrow\ 0 < u < 1 \text{ and } 0 < \frac{v}{\sqrt{u}} < 1 \ \Rightarrow\ 0 < v < \sqrt{u}.
$$

Note that the Jacobian is

$$
J(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ y & x \end{vmatrix} = 2x^2 \ \Rightarrow\ |J(u,v)| = \frac{1}{|J(x,y)|} = \frac{1}{2x^2} = \frac{1}{2u}.
$$

Now, since $f_{U,V}(u,v) = f_{X,Y}\big(x(u,v), y(u,v)\big)|J(u,v)|$, we have

$$f_{U,V}(u,v) = f_{X,Y}\left(\sqrt{u}, v/\sqrt{u}\right) \cdot \frac{1}{2u} = \begin{cases} \dfrac{2v}{u}, & \text{for } 0 < u < 1,\ 0 < v < \sqrt{u}, \\ 0, & \text{elsewhere.} \end{cases}$$

(b)(10pts) Using the joint density of (U,V), we have

$$P(U^2 < V) = P(0 < U < 1,\ U^2 < V < \sqrt{U}) = \int_0^1 \int_{u^2}^{\sqrt{u}} f_{U,V}(u,v)\, dvdu = \int_0^1 \int_{u^2}^{\sqrt{u}} \frac{2v}{u}\, dvdu$$

. Alternately, we have

$$P(U^2 < V) = P(V^2 < U < \sqrt{V},\ 0 < V < 1) = \int_0^1 \int_{v^2}^{\sqrt{v}} \frac{2v}{u}\, dudv.$$

(c)(10pts) The marginal density function of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_0^{\sqrt{u}} \frac{2v}{u}\, dv = \frac{v^2}{u}\bigg|_0^{\sqrt{u}} = \frac{(\sqrt{u})^2}{u} = 1 \text{ for } 0 < u < 1 \text{ and } 0 \text{ otherwise.}$$

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