APPM 3570/STAT 3100 — Exam 2 — Fall 2023

On the front of your bluebook, write (1) **your name**, (2) **Exam 2**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:____________________________ Date:_________

1. (40pts) There are 4 unrelated parts to this question. Justify your answers.

   (a) (10pts) An average of 3 car accidents occur per week on a stretch of highway. Find the probability that there will be 4 car accidents this week given that 1 accident has already occurred. Simplify your answer.

   (b) (10pts) A *Rayleigh* random variable, W, has cumulative distribution function $F(w) = 1 - e^{-bw^2/2}$ where $w \geq 0$ and $b > 0$ is a fixed constant (and $F(w) = 0$ if $w < 0$). Find the *hazard rate function* for W. (Do not try to find b.)

   (c) (10pts) Let $Z \sim \text{Normal}(\mu = 0, \sigma^2 = 1)$ and let $Y = Z^2$, find the *probability density function* of Y. Show all work.

   (d) (10pts) A bookshelf contains 8 distinct books, three are probability books, two are statistics books, and three are physics books. Two books are chosen at random. Let R be the number of probability books that are chosen and let T be the number of statistics books that are chosen, find the *joint probability mass function* of R and T.

Solution: (a)(10pts) Let X be the number of accidents that occur then $X \sim \text{Poisson}(\lambda = 3)$, now we wish to find $\mathrm{P}(X = 4 | X \geq 1)$, now

$$\mathrm{P}(X=4|X\geq 1) = \frac{\mathrm{P}(X=4, X\geq 1)}{\mathrm{P}(X\geq 1)} = \frac{\mathrm{P}(X=4)}{\mathrm{P}(X\geq 1)} = \frac{\mathrm{P}(X=4)}{1-\mathrm{P}(X<1)} = \frac{\mathrm{P}(X=4)}{1-\mathrm{P}(X=0)}$$

now note $p(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, so

$$\mathrm{P}(X=4|X\geq 1) = \frac{\mathrm{P}(X=4)}{1-\mathrm{P}(X=0)} = \frac{e^{-3}3^4/4!}{1-e^{-3}} = \frac{e^{-3}3^4}{4!(1-e^{-3})} = \frac{3^4}{4!(e^3-1)}.$$

(b)(10pts) Here we have

$$\lambda(w) = \frac{F'(w)}{1-F(w)} = \frac{-(-bwe^{-bw^2/2})}{1-(1-e^{-bw^2/2})} = \frac{bw\,\cancel{e^{-bw^2/2}}}{\cancel{e^{-bw^2/2}}} = bw \text{ for } w \geq 0.$$

(c)(10pts) Let $Z \sim \text{Normal}(\mu = 0, \sigma^2 = 1)$ and let $Y = Z^2$ then $Y \geq 0$ and the cdf of Y is

$$F_Y(y) = P(Y \leq y) = P(Z^2 \leq y) = P(-\sqrt{y} \leq Z \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

and recall that $\Phi'(z) = f_Z(z) = \dfrac{1}{\sqrt{2\pi}}e^{-z^2/2}$, $z \in \mathbb{R}$, so differentiating $F_Y(y)$ yields

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left[\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\right] = \frac{f_Z(\sqrt{y})}{2\sqrt{y}} - \frac{-f_Z(-\sqrt{y})}{2\sqrt{y}} = \frac{e^{-(\sqrt{y})^2/2} + e^{-(-\sqrt{y})^2/2}}{\sqrt{2\pi}\cdot 2\sqrt{y}}$$

so $f_Y(y) = \begin{cases} \dfrac{e^{-y/2}}{\sqrt{2\pi y}}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$ (This is known as the *chi-square* distribution.)

(d)(10pts) Note that here the order does not matter, thus the joint pmf is

$$P(R=0,T=0) = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28},\ P(R=1,T=0) = \frac{\binom{3}{1}\cdot\binom{3}{1}}{\binom{8}{2}} = \frac{9}{28},\ P(R=2,T=0) = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}$$

$$P(R=1,T=1) = \frac{\binom{3}{1}\cdot\binom{2}{1}}{\binom{8}{2}} = \frac{6}{28},\ P(R=0,T=1) = \frac{\binom{2}{1}\cdot\binom{3}{1}}{\binom{8}{2}} = \frac{6}{28},\ P(R=0,T=2) = \frac{\binom{2}{2}}{\binom{8}{2}} = \frac{1}{28},$$

and $P(R=r, T=t) = 0$ otherwise,

2. (30pts) It's time for Chip to file taxes for 2022 with the IRS! Chip is allowed to itemize deductions only if the total of all itemized deductions for 2022 is at least \$5,000. Let X (in 1000s of dollars) be Chip's total itemized deductions for 2022. Assume that X has the pdf

$$f(x) = \begin{cases} \frac{k}{x^4}, & \text{if } x \geq 5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (6pts) Find the value of the constant k.

(b) (6pts) What is the *cumulative distribution function* of X? The cdf should be defined for all real numbers.

(c) (6pts) What is Chip's expected total itemized deduction for 2022?

(d) (6pts) Find $E[X^2]$, then find $V(X)$.

(e) (6pts) Let $Y = \ln(X/5)$, find the *probability density function* for Y. The pdf should be defined for all real numbers.

Solution:

(a)(6pts) We need k such that

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_5^{\infty} kx^{-4}\,dx = \frac{-k}{3x^3}\bigg|_5^{\infty} = 0 + \frac{k}{3 \cdot 5^3} \quad \Rightarrow \quad \boxed{k = 3 \cdot 5^3 = 375.}$$

(b)(6pts) Note that, for $a \geq 5$, we have

$$F(a) = \mathrm{P}(X \leq a) = \int_{-\infty}^{a} f(x)\,dx = \int_5^a 375x^{-4}\,dx = -\frac{375}{3x^3}\bigg|_5^a = -\frac{125}{x^3}\bigg|_5^a = 1 - \frac{125}{a^3}$$

so, the cdf of X is

$$F_X(a) = \begin{cases} 0, & a < 5, \\ 1 - \frac{125}{a^3}, & a \geq 5. \end{cases}$$

(c)(6pts) Note that,

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_5^{\infty} x \cdot \frac{375}{x^4}\,dx = 375 \int_5^{\infty} x^{-3}\,dx = -\frac{375}{2x^2}\bigg|_5^{\infty} = \frac{375}{50} = \frac{15}{2}$$

so $\boxed{\text{Chip's expected total itemized deduction for 2022 is } \$7,500.}$

(d)(6pts) Here,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_5^{\infty} x^2 \cdot \frac{375}{x^4}\,dx = 375 \int_5^{\infty} x^{-2}\,dx = -\frac{375}{x}\bigg|_5^{\infty} = \frac{375}{5} = 75$$

and so

$$V[X] = E[X^2] - E[X]^2 = \frac{375}{5} - \left(\frac{375}{50}\right)^2 = 75 - \left(\frac{15}{2}\right)^2 = \frac{300}{4} - \frac{225}{4} = \boxed{\frac{75}{4}.}$$

(e)(6pts) We have,

$$F_Y(a) = \mathrm{P}(Y \leq a) = \mathrm{P}(\ln(X/5) \leq a) = \mathrm{P}(X/5 \leq e^a) = \mathrm{P}(X \leq 5e^a) = F_X(5e^a)$$

and, from part (b), we know that if $5e^a \geq 5$ then

$$F_X(5e^a) = 1 - \frac{125}{(5e^a)^3} = 1 - e^{-3a}$$

and also note that $5e^a \geq 5$ implies $a \geq 0$ and so,

$$F_Y(a) = 1 - e^{-3a} \Rightarrow f_Y(a) = 3e^{-3a} \text{ for } a \geq 0.$$

Thus, $\boxed{f_Y(y) = 3e^{-3y} \text{ for } y \geq 0 \text{ and } f_Y(y) = 0 \text{ otherwise,}}$ so we see $f_Y(y) \sim \mathrm{Exp}(\lambda = 3)$.

3. (30pts) Suppose that X, the price of a certain commodity (in dollars), and Y, its total sales (in 100 units), are random variables whose joint probability mass function can be approximated closely with the joint probability density

$$f(x,y) = \begin{cases} 8x(1-y), & \text{if } 0 < x < 1,\ 0 < y < 1-x, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (6pts) In your blue book make a sketch of the domain of $f(x,y)$ (*i.e.* sketch and shade the support of $f(x,y)$). Clearly label all axes, relevant points and functions otherwise points will be deducted.

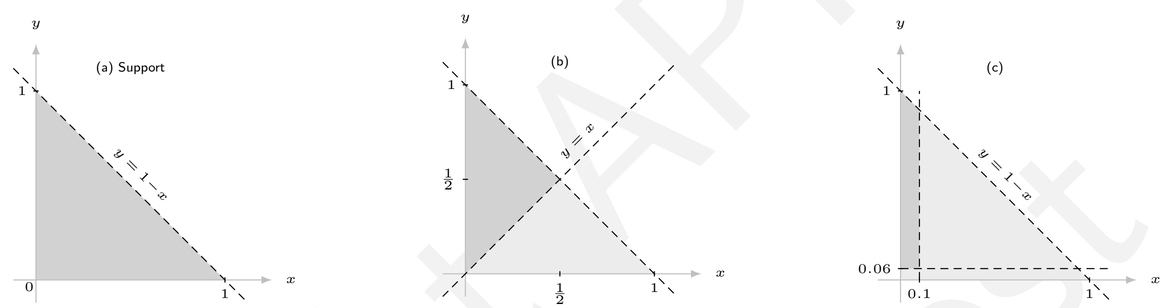
(b) (6pts) Find $\mathrm{P}(Y > X)$. Simplify your answer completely.

(c) (6pts) Set up, *but do not solve*, a double integral to find the probability that the price will be less than 10 cents and sales will exceed 6 units. (Check the units!)

(d) (6pts) Find the *marginal density function* of the random variable Y. The pdf should be defined for all real numbers.

(e) (6pts) Are the random variables X and Y *independent*? Why or why not?

Solution:



(a)(6pts) The support is the region enclosed by $x = 0$, $y = 0$ and $y = 1 - x$, see the sketch above.

(b)(6pts) Note that

$$\begin{aligned}
\mathrm{P}(Y > X) &= \int_0^{1/2} \int_x^{1-x} 8x(1-y)\ dy\ dx \\
&= \int_0^{1/2} 8x \left(y - \frac{y^2}{2} \right) \Bigg|_{y=x}^{y=1-x} dx \\
&= \int_0^{1/2} 8x \left[(1-x) - \frac{(1-x)^2}{2} - \left(x - \frac{x^2}{2} \right) \right] dx \\
&= \int_0^{1/2} 8x \left[\frac{1}{2} - x \right] dx \\
&= \int_0^{1/2} \left[4x - 8x^2 \right] dx = \left[2x^2 - \frac{8x^3}{3} \right]_0^{1/2} = \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{1}{6}.
\end{aligned}$$

Alternately, could also use $\mathrm{P}(Y > X) = \int_0^{1/2} \int_0^y 8x(1-y)\ dx\ dy + \int_{1/2}^1 \int_0^{1-y} 8x(1-y)\ dx\ dy.$

(c)(6pts) Here we have

$$\mathrm{P}(X < 0.1, Y > 0.06) = \int_0^{0.1} \int_{0.06}^{1-x} 8x(1-y)\ dy\ dx,$$

or $\mathrm{P}(X < 0.1, Y > 0.06) = \int_{0.06}^{0.9} \int_0^{0.1} 8x(1-y)\ dx\ dy + \int_{0.9}^{1} \int_0^{1-y} 8x(1-y)\ dx\ dy.$

(d)(6pts) The marginal density of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^{1-y} 8x(1-y)\, dx = 4x^2(1-y) \Big|_0^{1-y} = 4(1-y)^3,\ 0 < y < 1 \text{ and } f_Y(y) = 0 \text{ otherwise.}$$

(e)(6pts) No, X and Y are not independent since $\frac{f(x,y)}{f_Y(y)}$ is not a function of x only. If X and Y were independent, then $\frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$ and, so, if X and Y are independent then we expect $\frac{f(x,y)}{f_Y(y)}$ to be a function of x only. (Note that, for each $y \in (0,1)$ and x such that $0 < x < 1-y$, we have $\frac{f(x,y)}{f_Y(y)} = \frac{8x(1-y)}{4(1-y)^3} = \frac{2x}{(1-y)^2}$.)

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