APPM 3570/STAT 3100 — Exam 1 — Fall 2023

On the front of your bluebook, write (1) **your name**, (2) **Exam 1**, (3) **APPM 3570/STAT 3100**. Correct answers with no supporting work may receive little or no credit. Books, notes and electronic devices of any kind are not allowed. A sheet of interesting formulae accompanies this exam. Your exam should be uploaded to Gradescope in a PDF format (Recommended: **Genius Scan**, **Scannable** or **CamScanner** for iOS/Android). **Show all work, justify your answers.** **Do all problems.** Students are required to re-write the **honor code statement** in the box below on the **first page** of their exam submission and **sign and date it**:

On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work. Signature:____________________________ Date:_________

1. (32pts) There are 4 unrelated parts to this question. Justify your answers.

   (a) (8pts) How many different permutations are there of the 12 letters $A, B, C, \ldots, K, L$ for which (i) A and B are next to each other? (ii) A is before B (but not necessarily next to B)?

   (b) (8pts) A committee of 12 is to be selected from 10 seniors and 10 freshmen, in how many ways can the selection be carried out if there must be more freshmen than seniors?

   (c) (8pts) How many distinct terms are there in the complete expansion of $\left(\frac{x}{2}+y-3z\right)^5$? What is the sum of all the coefficients in the complete expansion?

   (d) (8pts) An urn contains M white and N black balls. If a random sample of size r is chosen, what is the probability that it contains exactly k white balls? (Assume that $k \le r$.)

Solution:

(a)(i)(5pts) We count the number of permutations that either contain the block AB or BA. If we consider AB to be a single "superletter" then we have a total of $12-2+1=11$ objects to permute, which can be done in $11!$ ways and, similarly, the number of permutations containing the block BA is also $11!$, thus, we have a total of $11!+11! = 2!\cdot 11!$ permutations where A and B are next to each other.

(a)(ii)(3pts) Let S be the set of all permutations of the 12 letters then $|S| = 12!$ and each one of these permutations must fall into one of two categories: either (i) A comes before B or (ii) B comes before A. Also note that there are just as many permutations of type (i) as there are of type (ii) since, for any permutation with A before B, if we switch these letters, then we get a permutation with B before A and vice-versa. Therefore we can conclude that we have a total of $\frac{12!}{2}$ permutations in which A is before B but not necessarily next to it.

(b)(8pts) This is the number of ways we can have 7, 8, 9, or 10 freshmen on the committee, so we have

$$\binom{10}{7}\binom{10}{5}+\binom{10}{8}\binom{10}{4}+\binom{10}{9}\binom{10}{3}+\binom{10}{10}\binom{10}{2} = \sum_{k=7}^{10}\binom{10}{k}\binom{10}{12-k} \text{ possible committees.}$$

(c)(i)(4pts) Each term has the form $x^{n_1}y^{n_2}z^{n_3}$ where $n_1+n_2+n_3=5$ for non-negative integers n_1, n_2 and n_3. By a theorem in class, the number of non-negative integer solutions of $n_1+n_2+n_3=5$ is $\binom{5+3-1}{3-1}$ so there are $\binom{7}{2}$ terms.

(c)(ii)(4pts) Note that, by the Multinomial Theorem, we have

$$\left(\frac{x}{2}+y-3z\right)^5 = \sum_{n_1+n_2+n_3=5}\binom{5}{n_1,n_2,n_3}(1/2)^{n_1}(1)^{n_2}(-3)^{n_3}\cdot x^{n_1}y^{n_2}z^{n_3}$$

so to find the sum of all the coefficients in the expansion we need to make the variables vanish, so let $x=1$, $y=1$ and $z=1$, thus the sum of the coefficients of the complete expansion is

$$\sum_{n_1+n_2+n_3=5}\binom{5}{n_1,n_2,n_3}(1/2)^{n_1}(1)^{n_2}(-3)^{n_3} = \left(\frac{1}{2}+1-3\right)^5 = \left(-\frac{3}{2}\right)^5$$

(d)(8pts) First note that the urn contains a total of $M+N$ balls and the size of the sample space of this experiment is $\binom{M+N}{r}$. Consider the event that k white balls are selected (we assume $k\le r$) then the remaining $r-k$ balls selected are black, thus

$$P(\text{selecting exactly } k \text{ white balls}) = \frac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}.$$

2. (32pts) You have 6 coins in front of you. Five of the coins are unbiased (i.e., the probability of tossing a head is 50%). The sixth coin is biased, and the probability of tossing a head is 70%. It is not possible to tell which is the biased coin just by looking. You plan to pick a coin at random and then will flip it three times.

   (a) (4pts) If an unbiased coin was picked, what is the probability that two of three tosses will be heads? What assumptions are you making?

   (b) (4pts) If the biased coin was picked, what is the probability that two of three tosses will be heads? What assumptions are you making?

   (c) (8pts) What is the total probability of tossing two heads?

   (d) (8pts) You pick a coin, toss it three times and get two heads. What is the probability you selected the biased coin?

   (e) (8pts) Now let Y be the *total number of flips of the biased coin required to get exactly three heads* (we stop flipping after the occurrence of the third head). Find $P(Y = n)$ for $n = 3, 4, \dots$.

Solution: Let X be the number of heads observed and let F be the event that we select a fair coin and let $F^c = B$ be the event that we select a bias coin.

(a)(4pts) Note we need to take account the number of ways the event can happen and the probability of the event so, in the event that we choose the *fair* coin, the probability of getting two heads out of three is

$$P(X = 2 \mid F) = \binom{3}{2} p^2(1-p) = \binom{3}{2}(0.5)^2(1-0.5) = \binom{3}{2}(0.5)^3 = \underbrace{\frac{3}{8} = 0.375.}_{\text{for grading purposes}}$$

Since the coin is fair, we can calculate this probability by using counting or, alternately, we can calculate the probability by assuming each coin flip is independent.

(b)(4pts) Similarly, if we pick the bias coin, we have

$$P(X = 2 \mid B) = \binom{3}{2} p^2(1-p) = \binom{3}{2}(0.7)^2(1-0.7) = \binom{3}{2}(0.7)^2(0.3) = \underbrace{0.441.}_{\text{for grading purposes}}$$

We assume that each coin flip is independent of the other in calculating this probability.

(c)(8pts) Using the law of total probability, conditioning, and the fact that $F^c = B$, yields

$$\begin{aligned}
P(X = 2) &= P\Big((\{X = 2\} \cap F) \cup (\{X = 2\} \cap F^c)\Big) \\
&= P(\{X = 2\} \cap F) + P(\{X = 2\} \cap B) \\
&= P(X = 2 \mid F)P(F) + P(X = 2 \mid B)P(B) \\
&= \binom{3}{2}(0.5)^2(0.5) \times \frac{5}{6} + \binom{3}{2}(0.7)^2(0.3) \times \frac{1}{6} = \underbrace{0.386.}_{\text{for grading purposes}}
\end{aligned}$$

(d)(8pts) Finally, note that, by Bayes Rule, we have

$$P(B|X = 2) = \frac{P(B \cap \{X = 2\})}{P(X = 2)} = \frac{P(X = 2 \mid B)P(B)}{P(X = 2)} = \frac{\binom{3}{2}(0.7)^2(0.3) \times \frac{1}{6}}{\binom{3}{2}(0.5)^2(0.5) \times \frac{5}{6} + \binom{3}{2}(0.7)^2(0.3) \times \frac{1}{6}} = \underbrace{0.19.}_{\text{for grading purposes}}$$

(e)(8pts) We need to flip the coin at least three times to observe three heads so $Y \geq 3$. The event $Y = n$ implies that the third head occurred on the n-th flip (with probability $p = 0.7$) and two heads occurred in the first $n-1$ flips, which can happen in $\binom{n-1}{2}$ ways and with probability $p^2(1-p)^{(n-1)-2}$, thus, we have

$$P(Y = n) = \binom{n-1}{2} p^2(1-p)^{n-3} \cdot p = \binom{n-1}{2}(0.7)^3(0.3)^{n-3} \quad \text{for } n = 3, 4, \dots.$$

3. (36pts) Shedeur and Shiloh are playing a game with two 3-sided dice, die A and die B (note, 3-sided dice do not exist but, for the sake of this question, let's just say they do). Die A has the values -1, 0, and 1, and is biased, with $\mathrm{P}(A = -1) = \frac{2}{10}$, $\mathrm{P}(A = 0) = \frac{4}{10}$ and $\mathrm{P}(A = 1) = \frac{4}{10}$, and Die B has the values 1, 2, 3 and is a fair die (so all sides of Die B are equiprobable).

   The game works as follows: both dice are rolled, and Shiloh pays Shedeur the product of the amounts shown on the faces of Die A and Die B, if the product is negative, Shedeur pays Shiloh. (For example, if Die A=1 and Die B=3, then the product is $1 \cdot 3 = 3$ and Shiloh has to pay Shedeur $\$3$; if Die A=-1 and Die B=2 then the product is $-1 \cdot 2 = -2$ and Shedeur has to pay Shiloh $\$2$.)

(a) (8pts) What is the sample space of all possible outcomes of rolling the pair of dice? (Your answer should be in the appropriate set notation.)

(b) (4pts) Define the random variable X to be the amount of money Shedeur gains/loses on a single turn. What values can X take on?

(c) (8pts) Find the *probability mass function* (pmf) of X. What assumptions are you making about dice A and B? Verify that your answer is a true probability mass function. (Your pmf should be defined for all real numbers.)

(d) (8pts) Write down the *cumulative distribution function* (cdf) of X. (Your cdf should be defined for all real numbers.)

(e) (8pts) Find the probability that Shedeur wins at least two dollars.

Solution:

(a)(8pts) $\mathcal{S} = \{(x_A, x_B) | x_A \in \{-1, 0, 1\},\ x_B \in \{1, 2, 3\}\} = \{(-1, 1), (0, 1), (1, 1), (-1, 2), (0, 2), (1, 2), (-1, 3), (0, 3), (1, 3)\}$.

(b)(4pts) $X((x_A, x_B)) = x_A \cdot x_B$=Shedeur's gains/losses and so $X \in \{-3, -2, -1, 0, 1, 2, 3\}$.

(c)(8pts) We assume the outcomes of dice A and of dice B are **independent outcomes**. Thus the pmf of X is,

$$\begin{aligned}
p(-3) &= \mathrm{P}(X = -3) = \mathrm{P}(\{(x_A, x_B) = (-1, 3)\}) = \frac{2}{10} \cdot \frac{1}{3} = \frac{2}{30} \\
p(-2) &= \mathrm{P}(\{(-1, 2)\}) = \frac{2}{30} \\
p(-1) &= \mathrm{P}(\{(-1, 1)\}) = \frac{2}{30} \\
p(0) &= \mathrm{P}(\{(0, 1)\}) + \mathrm{P}(\{(0, 2)\}) + \mathrm{P}(\{(0, 3)\}) = 3\left(\frac{4}{10} \cdot \frac{1}{3}\right) = \frac{2}{5} \\
p(1) &= \mathrm{P}(\{(1, 1)\}) = \frac{4}{10} \cdot \frac{1}{3} = \frac{4}{30} \\
p(2) &= \mathrm{P}(\{(1, 2)\}) = \frac{4}{30} \\
p(3) &= \mathrm{P}(\{(1, 3)\}) = \frac{4}{30} \\
p(k) &= 0 \text{ for } k \notin \{-3, -2, -1, 0, 1, 2, 3\}
\end{aligned}$$

We can verify this is indeed a pmf, note that $0 \leq p(k) \leq 1$ and

$$\sum_{k=-3}^{3} p(k) = 3 \cdot \left(\frac{2}{30}\right) + \frac{12}{30} + 3 \cdot \left(\frac{4}{30}\right) = \frac{6 + 12 + 12}{30} = 1.$$

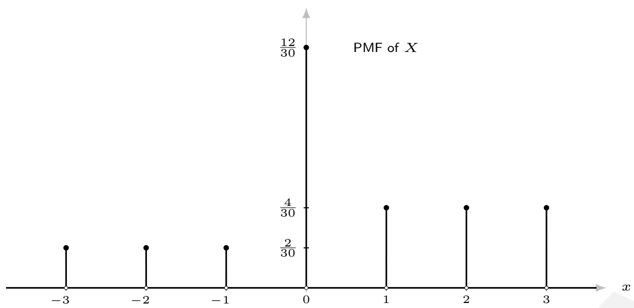
(d)(8pts) The cdf can be written as

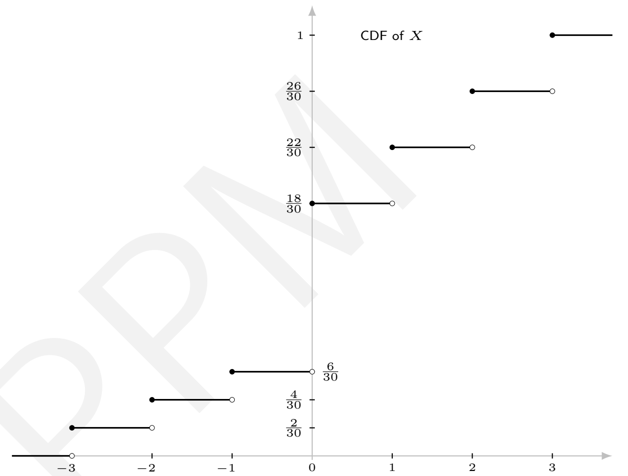
$$F(x) = \begin{cases} 0, & \text{if } x < -3, \\ \frac{2}{30}, & \text{if } -3 \leq x < -2, \\ \frac{4}{30}, & \text{if } -2 \leq x < -1, \\ \frac{6}{30}, & \text{if } -1 \leq x < 0, \\ \frac{18}{30}, & \text{if } 0 \leq x < 1, \\ \frac{22}{30}, & \text{if } 1 \leq x < 2, \\ \frac{26}{30}, & \text{if } 2 \leq x < 3, \\ 1, & \text{if } x \geq 3. \end{cases}$$

(e)(8pts) We wish to find the probability that Shedeur wins \$2 or \$3, so,

$$\mathrm{P}(X \geq 2) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3) = p(2) + p(3) = \frac{4}{30} + \frac{4}{30} = \frac{8}{30} = \frac{4}{15}.$$

Just for fun, the plot of the pmf and cdf are shown below:





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