APPM 3570/STAT 3100 Exam 3 Fall 2022

NAME: ______________________________

SECTION: 001 at 9:05 am

Instructions:

1. Notes, your text and other books, cell phones, and other electronic devices are not permitted, except as needed to view and upload your work, and except calculators.
2. Calculators are permitted.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope. Verify that everything has been uploaded correctly and the pages have been associated with the correct problems.
5. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________ Date: ____________

Problem 1. (28 points) Parts (a), (b) and (c) are unrelated.

(a) Suppose that 3 marbles are chosen without replacement from an urn containing 8 green and 2 yellow marbles. Let X_i equal 1 if the ith marble selected is yellow and let it equal 0 otherwise, for $i = 1, 2, 3$.

  (i) Find $\mathrm{P}(X_3 = 1 \mid X_1 = 1)$.

  (ii) Find $\mathrm{P}(X_1 = X_3)$.

(b) The number of calories in a cheeseburger on the lunch menu is approximately normally distributed with a mean of 434 and a variance of 49.

  (i) What is the probability that a randomly chosen cheeseburger will contain more than 420 calories?

  (ii) Alex orders 8 cheeseburgers for a party. Assuming independence, find the probability that the total calories in the 8 cheeseburgers will exceed 3,450.

  (iii) If the 8 cheeseburgers are served one at a time to 8 guests, what is the probability that the first guest to be served a cheeseburger with over 420 calories is the seventh guest? Explain.

(c) Suppose the number of foreign fragments in a portion of peanut butter is a random variable with a mean of 3 and a variance of 3. Suppose a random sample of 50 portions of peanut butter are collected and the average number of foreign fragments in a portion is calculated. What is the probability of observing an average of 3.6 or more fragments?

Solution:

(a) (10 points)

(i) (5 points) The possible outcomes of X_1 and X_3 are:

Outcome	X_1	X_3	Prob.
YYG	1	0	$2/10 * 1/9 * 8/8 = 16/720$
YGY	1	1	$2/10 * 8/9 * 1/8 = 16/720$
YGG	1	0	$2/10 * 8/9 * 7/8 = 112/720$
GYY	0	1	$8/10 * 2/9 * 1/8 = 16/720$
GYG	0	0	$8/10 * 2/9 * 7/8 = 112/720$
GGY	0	1	$8/10 * 7/9 * 2/8 = 112/720$
GGG	0	0	$8/10 * 7/9 * 6/8 = 336/720$

The joint pmf of X_1 and X_3 is:

		X_3 = 0	X_3 = 1
X_1	0	448/720	128/720
	1	128/720	16/720

The joint pmf of X_1 and X_3 after simplification is:

		X_3 = 0	X_3 = 1
X_1	0	28/45	8/45
	1	8/45	1/45

$\mathrm{P}(X_3 = 1 \,|\, X_1 = 1) = \frac{\mathrm{P}(X_3=1, X_1=1)}{\mathrm{P}(X_3=1)} = \frac{\frac{1}{45}}{\frac{9}{45}} = \frac{1}{9}$.

(ii) (5 points) $\mathrm{P}(X_1 = X_3) = \mathrm{P}(X_1 = 0,\, X_3 = 0) + \mathrm{P}(X_1 = 1,\, X_3 = 1) = \frac{28}{45} + \frac{1}{45} = \frac{29}{45}$.

(b) (12 points)

(i) (4 points) We are given that $X_i \sim \mathcal{N}(434, \sigma^2 = 49)$, for $i = 1, 2, \ldots$. Therefore, $\mathrm{P}(X_i > 420) = \mathrm{P}(Z > \frac{420-434}{7}) = \mathrm{P}(Z > -2) = \Phi(2) \approx .9772$.

(ii) (4 points) Let $Y = \sum_1^8 X_i$ be the total calories in eight cheeseburgers. Then, assuming independence of X_i, we have $Y \sim \mathcal{N}(3472, \sigma^2 = 392)$ and $\mathrm{P}(Y > 3450) = \mathrm{P}(Z > \frac{3450-3472}{19.799}) \approx \mathrm{P}(Z > -1.11) = \Phi(1.11) = .8665$.

(iii) (4 points) Let S be the number of servings required until the first time a guest is served a cheeseburger with over 420 calories, then $S \sim Geom(.9772)$. $\mathrm{P}(S = 7) = (1 - .9722)^6(.9772) \approx 1.37 * 10^{-10} \approx 0$.

(c) (6 points) By the Central Limit Theorem, $\overline{X} \overset{approx}{\sim} \mathcal{N}(3, \sigma^2 = \frac{3}{50})$.
$\mathrm{P}(X > 3.6) = \mathrm{P}(Z > \frac{3.6-3}{.245}) \approx \mathrm{P}(Z > 2.45) = 1 - \Phi(2.45) \approx 1 - .9929 = .0071$.

Problem 2. (28 points) Let (X, Y) be jointly distributed random variables with conditional pdf given by:

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{x^2} & 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

and marginal pdf of X given by:

$$f_X(x) = \begin{cases} 5x^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint pdf of X and Y.

(b) Find the marginal pdf of Y.

(c) Find $\mathrm{E}[Y|X]$ and use it to find the expectation of Y.

(d) Find $\mathrm{Cov}(X, Y)$.

Solution:

(a) (7 points)

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x)$$

$$f_{X,Y}(x, y) = \begin{cases} 10x^2y & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) (7 points)

$$\begin{aligned} f_Y(y) &= \int_y^1 10x^2y \; dx \\ &= \frac{10x^3y}{3}\Big|_y^1 \\ &= \frac{10y}{3} - \frac{10y^4}{3} \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{10y}{3} - \frac{10y^4}{3} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) (7 points)

$$\begin{aligned}
\text{E}[Y|X] &= \int_0^x \frac{2y^2}{x^2}\ dy \\
&= \frac{2y^3}{3x^2}\Big|_0^x \\
&= \frac{2x}{3}
\end{aligned}$$

$$\begin{aligned}
\text{E}[Y] = \text{E}[\text{E}[Y|X]] &= \int_0^1 \frac{2x}{3} 5x^4\ dx \\
&= \frac{10x^6}{18}\Big|_0^1 \\
&= \frac{10}{18} = \frac{5}{9}
\end{aligned}$$

(d) (7 points)

$$\begin{aligned}
\text{E}[XY] &= \int_0^1 \int_y^1 10x^3y^2\ dx\, dy \\
&= \int_0^1 \frac{10x^4y^2}{4}\Big|_y^1\ dy \\
&= \int_0^1 \frac{10y^2}{4} - \frac{10y^6}{4}\ dy \\
&= \frac{5y^3}{6} - \frac{5y^7}{14}\Big|_0^1 \\
&= \frac{5}{6} - \frac{5}{14} = \frac{35}{42} - \frac{15}{42} = \frac{20}{42} \\
&= \frac{10}{21}
\end{aligned}$$

$$\begin{aligned}
\text{E}[X] &= \int_0^1 5x^5\ dx \\
&= \frac{5x^6}{6}\Big|_0^1 \\
&= \frac{5}{6}
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(X,Y) &= \text{E}[XY] - \text{E}[X]\text{E}[Y] \\
&= \frac{10}{21} - \left(\frac{5}{6}\right)\left(\frac{5}{9}\right) \\
&= \frac{10}{21} - \frac{25}{54} \\
&= \frac{5}{378} \approx .0132
\end{aligned}$$

Problem 3. (18 points) If X and Y have the following joint pdf, compute the joint density of $U = \frac{X}{Y},\ V = Y$.

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

(6 points)

$$X = UV$$
$$Y = V$$

(6 points)

$$\text{J} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}$$

$$= v$$
$$|\text{J}| = |v| = v$$

(6 points)

$$f_{U,V}(u,v) = \begin{cases} 2v & 0 < u < 1, \quad 0 < v < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 4. (26 points) Recall that if $X \sim \text{Exp}(\lambda)$, then $\text{E}[X] = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$.
Recall that if $Y \sim \text{Unif}(0,1)$, then $\text{E}[Y] = 1/2$ and $\text{Var}(Y) = 1/12$.
Mason has 200 batteries whose lifetimes are independent exponential random variables, each with a mean of 4 hours.

(a) If the batteries are used one at a time, with a failed battery being replaced immediately by a new one, approximate the probability that there is still a working battery after 810 hours.

(b) Suppose that the time it takes (in hours) to replace a failed battery is uniformly distributed over (0, .8), independently. Approximate the probability that all batteries have failed before 1000 hours have passed.

Solution:

(a) (11 points)
Let X_i be the lifetime of the ith battery, $i = 1, 2, \ldots, 200$.
$\mathrm{E}[X_i] = 4,\ \mathrm{Var}(X_i) = 16$
By the CLT, $\sum_{n=1}^{200} X_i \overset{approx}{\sim} \mathcal{N}(800, 3200)$

$$\begin{aligned} \mathrm{P}(\sum_{n=1}^{200} X_i > 810) &\approx \mathrm{P}\Big(Z > \frac{810 - 800}{\sqrt{3200}}\Big) \\ &= \mathrm{P}(Z > .1768) \\ &\approx 1 - \Phi(.18) \\ &= 1 - .5714 = .4286 \end{aligned}$$

(b) (15 points)
Let R_i be the time needed to replace the ith battery, $i = 1, 2, \ldots, 200$.
$\mathrm{E}[R_i] = .4,\ \mathrm{Var}(R_i) = .8^2 * 1/12 = .0533$

$E[\sum_{n=1}^{199} R_i] = 79.6,\ \mathrm{Var}(\sum_{n=1}^{199} R_i) = 10.6067$

By the CLT, $\sum_{n=1}^{200} X_i + \sum_{n=1}^{199} R_i \overset{approx}{\sim} \mathcal{N}(879.6,\ 3210.61)$

$$\begin{aligned} \mathrm{P}(\sum_{n=1}^{200} X_i + \sum_{n=1}^{199} R_i < 1000) &\approx \mathrm{P}\Big(Z < \frac{1000 - 879.6}{\sqrt{3210.61}}\Big) \\ &= \mathrm{P}(Z < 2.1249) \\ &\approx \Phi(2.12) \\ &= .983 \end{aligned}$$

Bonus Problem. (3 points) Let X be a random variable with mean 50 and variance 25. If X is the number of cars produced in a week at a particular auto manufacturing plant, find a lower bound on the probability that the production of cars in a week is between 30 and 70.

Solution: $\mathrm{P}\big(|X - 50| \geq 20\big) \leq \frac{\mathrm{Var}(X)}{400} = \frac{25}{400} = \frac{1}{16}$.
$\mathrm{P}\big(|X - 50| < 20\big) \geq 1 - \frac{1}{16} = \frac{15}{16}$.