# APPM 3570/STAT 3100 Exam 2 Fall 2022

NAME: ____________________________________________

SECTION: 001 at 9:05 am $\boxed{or}$ 003 at 12:20 pm

**Instructions:**

1. Notes, your text and other books, cell phones, and other electronic devices are not permitted, except as needed to view and upload your work, and except calculators.
2. Calculators are permitted.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope. Verify that everything has been uploaded correctly and the pages have been associated with the correct problems.
5. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ____________________________ Date: ______________

**Problem 1.** (29 points) There are 3 unrelated parts to this question.

(a) Suppose you play a board game and you land in "jail" and the only way to get out of "jail" is to roll doubles using two fair six-sided dice (regular dice).

(i) What is the probability that it takes exactly 3 attempts (3 rolls of the two dice) to get out of "jail"?

(ii) What is the expected number of attempts needed to get out of "jail"?

(iii) What is the probability that it takes more than the expected number of attempts before you get out of "jail"?

(b) The life, in years, of a certain type of electrical switch has an exponential distribution with an average life of 2 years. What is the probability it fails during the first year?

(c) Let $X$ be a random variable and $\mathrm{P}(X = 1) = 1 - \mathrm{P}(X = 0)$. If $5\mathrm{Var}(X) = \mathrm{E}[X]$, find $\mathrm{P}(X = 0)$.

**Solution:**

(a) (15 points)

(i) The probability of rolling doubles on one roll of two fair dice is $\frac{1}{6}$. If $X$ is the number of rolls until a double appears, then $X \sim \text{Geom}(p = \frac{1}{6})$. Then $\text{P}(X = 3) = (\frac{5}{6})^2(\frac{1}{6}) \approx .1157$.

(ii) $\text{E}[X] = \frac{1}{p} = 6$ attempts

(iii) $\text{P}(X > 6) = \frac{(\frac{5}{6})^6(\frac{1}{6})}{1-\frac{5}{6}} = (\frac{5}{6})^6 \approx .3349$

(b) (7 points)

$$\begin{aligned}
\text{P}(X < 1) &= \int_0^1 \frac{1}{2}e^{-\frac{x}{2}}\,dx \\
&= -e^{-\frac{x}{2}}\Big|_0^1 \\
&= -e^{-\frac{1}{2}} + 1 \\
&= 1 - e^{-\frac{1}{2}} \\
&\approx .393
\end{aligned}$$

(c) (7 points) Let $p = \text{P}(X = 1)$. $E[X] = p$. $E[X^2] = p$ Therefore $\text{Var}(X) = p - p^2$. If $5\text{Var}(X) = E[X]$, then $5(p - p^2) = p$ which implies $p = \frac{4}{5}$ and $\text{P}(X = 0) = \frac{1}{5}$.

**Problem 2.** (31 points) A game consists of choosing two balls (without replacement) randomly from an urn containing 6 white balls, 3 red balls, and 1 yellow ball. Suppose that a person wins \$3 for each red ball selected and loses \$2 for each yellow ball selected. Let $X$ denote the winnings of the game.

(a) Find the probability mass function of $X$.

(b) Calculate the expected value of $X$.

(c) In the same game, let $Y$ equal the number of white balls selected. Write the joint probability mass function of $X$ and $Y$.

(d) Find the conditional probability mass function of $X$ given that $Y = 1$.

(e) Are $X$ and $Y$ independent? Justify your answer.

**Solution:**

(a) (7 points)

$$P(W\,W) = \frac{\binom{6}{2}}{\binom{10}{2}} = \frac{15}{45} \qquad X = 0$$

$$P(W\,R) = \frac{\binom{6}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{18}{45} \qquad X = 3$$

$$P(W\,Y) = \frac{\binom{6}{1}\binom{1}{1}}{\binom{10}{2}} = \frac{6}{45} \qquad X = -2$$

$$P(R\,Y) = \frac{\binom{3}{1}\binom{1}{1}}{\binom{10}{2}} = \frac{3}{45} \qquad X = 1$$

$$P(R\,R) = \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{3}{45} \qquad X = 6$$

(b) (7 points)

$$E[X] = \frac{54 - 12 + 3 + 18}{45} = \frac{63}{45} = \$1.40$$

(c) (7 points)

| x \ y | 0 | 1 | 2 |
|---|---|---|---|
| -2 | 0 | 6/45 | 0 |
| 0 | 0 | 0 | 15/45 |
| 1 | 3/45 | 0 | 0 |
| 3 | 0 | 18/45 | 0 |
| 6 | 3/45 | 0 | 0 |

(d) (5 points)

| $x\|Y=1$ | $P(x\|Y=1)$ |
|---|---|
| -2 | 1/4 |
| 3 | 3/4 |

(e) (5 points) No, $X$ and $Y$ are not independent. Their joint pmf is not the product of their marginal pmf's for every $(x, y)$. For example, $P(X = -2, Y = 0) \neq P(X = -2)P(Y = 0)$
since $P(X = -2, Y = 0) = 0$ and $P(X = -2)P(Y = 0) = 6/45 * 6/45$.

**Problem 3.** (24 points) Let $X$ and $Y$ be a random variables with joint probability density function:

$$f_{X,Y}(x, y) = \begin{cases} 3(x + y) & 0 < x < 1, \; 0 < y < 1 - x \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal probability density function of $Y$.

(b) Find $E[Y]$.

(c) Are $X$ and $Y$ independent? Justify your answer.

(d) Write the specific expression for the probability that $X > 2Y$, complete with any integration symbols and bounds that are needed, but do not solve.

**Solution:**

(a) (7 points)

$$
\begin{aligned}
f_Y(y) &= \int_0^{1-y} (3x + 3y)\, dx \\
&= \left( \frac{3x^2}{2} + 3xy \right) \Big|_0^{1-y} \\
&= \frac{3}{2}(1-y)^2 + 3y(1-y) \\
&= \frac{3 - 6y + 3y^2}{2} + \frac{6y - 6y^2}{2} \\
&= \frac{3}{2}(1 - y^2)
\end{aligned}
$$

$$
f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}
$$

(b) (7 points)

$$
\begin{aligned}
\mathrm{E}[Y] &= \int_0^1 \frac{3y}{2}(1-y^2)\, dy \\
&= \left( \frac{3y^2}{4} - \frac{3y^4}{8} \right) \Big|_0^1 \\
&= \frac{3}{4} - \frac{3}{8} \\
&= \frac{3}{8}
\end{aligned}
$$

(c) (5 points) No, $X$ and $Y$ are not independent. Their joint pdf cannot be factored into the product of two functions, one depending only on $X$ and one depending only on $Y$. Also, the non-rectangular region of integration implies that $X$ and $Y$ are not independent.

(d) (5 points)

$$
\mathrm{P}(X > 2Y) = \int_0^{1/3} \int_{2y}^{1-y} (3x + 3y)\, dx\, dy
$$

or equivalently,

$$
\mathrm{P}(X > 2Y) = \int_0^{2/3} \int_0^{x/2} (3x + 3y)\, dy\, dx + \int_{2/3}^{1} \int_0^{1-x} (3x + 3y)\, dy\, dx
$$

**Problem 4.** (16 points) A hiker regularly makes trail mix by combining cashews and raisins in a ziplock bag. The weight of cashews in each bag is normally distributed with a mean of 132 grams and a variance of 49 grams.

The amount of raisins in each bag is normally distributed with a mean of 60 grams and a variance of 9 grams.

Assume the amounts of cashews and raisins in a bag are independent of each other. Also assume the amounts of trail mix in each bag are independent.

(a) Find the $75th$ percentile of the weight of cashews in a bag. (Find the weight such that 75% of bags have a cashew weight below it.)

(b) Find the probability that the total weight of two bags of trail mix exceeds 368 grams.

**Solution:**

(a) (8 points) Let $X$ be the weight of cashews in a bag. $X \sim \mathcal{N}(132,\, 49)$

$$\begin{aligned} \mathrm{P}(Z < .67) &\approx .75 \quad \text{where } Z \sim \mathcal{N}(0,\, 1) \\ \frac{X-\mu}{\sigma} &= .67 \\ \frac{X-132}{7} &= .67 \\ X &= .67 * 7 + 132 \\ X &= 136.69 \end{aligned}$$

The $75th$ percentile of the weight of cashews in a bag is 136.69 grams.

(b) (8 points) Let $T_1$ be the total weight of cashews and raisins in a bag. Since the amounts of cashews and raisins in a bag are independent, $T_1 \sim \mathcal{N}(192,\, 58)$. Let $T_2$ be the total weight of cashews and raisins in a second bag, $T_2 \sim \mathcal{N}(192,\, 58)$. Since $T_1$ and $T_2$ are independent, $T_1 + T_2 \sim \mathcal{N}(384,\, 116)$.

$$\begin{aligned} \mathrm{P}(T_1 + T_2 > 368) &= \mathrm{P}\left(Z > \frac{368-384}{\sqrt{116}}\right) \quad \text{where } Z \sim \mathcal{N}(0,\, 1) \\ &\approx \mathrm{P}(Z > -1.486) \\ &\approx \mathrm{P}(Z < 1.49) \quad \text{using the standard normal table} \\ &\approx .9319 \end{aligned}$$

**Bonus Problem.** (3 points) What is the probability of observing exactly two 3's in four independent observations of a Poisson random variable with mean 5?

Recall that a Poisson random variable with parameter $\lambda > 0$ has the probability mass function: $\mathrm{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots.$

**Solution:** Let $X$ be the outcome of the Poisson random variable with mean 5. $\mathrm{P}(X = 3) = \frac{e^{-5}5^3}{3!} \approx .14$.
Let $Y$ be the number of 3's observed in 4 independent observations of $X$. $Y \sim \mathit{Binomial}(n = 4, p = .14)$. Then $\mathrm{P}(Y = 2) = \binom{4}{2}.14^2.86^2 \approx .087$.