APPM 3570/STAT 3100 Exam 1 Fall 2022

NAME: ______________________________

SECTION: 001 <u>or</u> 003

Instructions:

1. Notes, your text and other books, cell phones, and other electronic devices are not permitted, except as needed to view and upload your work, and except calculators.
2. Calculators are permitted.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope. Verify that everything has been uploaded correctly and the pages have been associated to the correct problems.
5. Turn in your hardcopy exam.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ______________________ Date: ____________

Problem 1. (24 points) There are 4 unrelated parts to this question.

(a) There are six parking spaces in a row. How many ways can 5 distinct cars park in the six spaces? (Assume only one car can park in a parking space and every car parks head-in.)

(b) There are twelve teams. Each team will play each other twice in the season. How many games will be played in the season?

(c) Ten people (Alex and Jose and 8 other people) are available to serve on a committee of six people. How many committees can be formed if Alex and Jose must be on the committee?

(d) How many arrangements can be formed from the letters in PEPPER?

Solution:

(a) (6 points) $6! = 720$

(b) (6 points) $\binom{12}{2} \cdot 2 = 66 \cdot 2 = 132$ games

(c) (6 points) $\binom{8}{4} = 70$ committees

(d) (6 points) $\binom{6}{3,2} = \frac{6!}{3!\,2!} = 60$ arrangements

Problem 2. (28 points) A University library has thirteen laptops available for student use. Three are considered good for creators C, six are designed for gaming G, and four are very light in weight L. During a heavy blizzard, a dorm resident hires a snowplow to pick up four laptops from the library for students to use at the dorm. The librarian randomly selects the four laptops.

(a) What is the probability that the dorm gets exactly two gaming laptops?

(b) What is the probability that the dorm gets at least one laptop of each type?

(c) What is the probability that the dorm gets two gaming laptops, a creator laptop and a light laptop?

(d) Use the inclusion/exclusion identity to compute the probability that the dorm gets two gaming laptops (exactly 2G), at least one of each type of laptop (at least one of C, G and L), or both events occur.

Solution:

(a) (6 points) $\mathrm{P}(\{\text{exactly } 2\, G\}) = \binom{6}{2}\binom{7}{2}/\binom{13}{4} = 315/715 = 63/143$

(b) (8 points) $\mathrm{P}(\{\text{at least one of each } C, G \text{ and } L\}) = \left[\binom{3}{2}\binom{6}{1}\binom{4}{1}+\binom{3}{1}\binom{6}{2}\binom{4}{1}+\binom{3}{1}\binom{6}{1}\binom{4}{2}\right]/\binom{13}{4} = 360/715 = 72/143$

(c) (6 points) $\mathrm{P}(\{2G, C, L\}) = \binom{3}{1}\binom{6}{2}\binom{4}{1}/\binom{13}{4} = 180/715 = 36/143$

(d) (8 points) $\mathrm{P}(\{\text{exactly } 2\, G) \cup \{\text{at least one of each } C, G \text{ and } L\})$
$= \mathrm{P}(\{\text{exactly } 2\, G\}) + P(\{\text{at least one of each } C, G \text{ and } L\})$
$- \mathrm{P}(\{\text{exactly } 2\, G) \cap \{\text{at least one of each } C, G \text{ and } L\}) = (63 + 72 - 36)/143 = 99/143$

Problem 3. (28 points) A blood test is developed for a viral disease and gives either a positive or negative result. The probability that a person has the disease is 0.1. Given that a person has the disease, the probability of a positive test result is 0.8. Given that a person does not have the disease, the probability of a negative test result is 0.8.

(a) If a person is randomly selected to take the blood test, what is the probability of a positive test result?

(b) Given a person tests positive, what is the probability that they have the disease?

(c) A second test is administered on saliva, independent of the blood test and gives either a positive or negative result. Given that a person has the disease, the probability of a positive saliva test result is 0.7. Given that a person does not have the disease, the probability of a negative saliva test result is 0.7. Given that a person tests positive on both tests, what is the probability that they have the disease?

(d) If three people are randomly selected, what is the probability that at least one of them has the disease?

Solution:

(a) (6 points) Let B be the event that the blood test is positive. $\mathrm{P}(B) = (.1)(.8)+(.9)(.2) = .26$

(b) (6 points) Let H be the event that the person has the disease. $\mathrm{P}(H|B) = (.1)(.8)/\mathrm{P}(B) = .08/.26 = 4/13 \approx .3077$

(c) (8 points) Let S be the event that the saliva test is positive. $\mathrm{P}(H|BS) = (.1)(.8)(.7)/[(.1)(.8)(.7)+(.9)(.2)(.3)] = .056/.11 \approx .5091$

(d) (8 points) $\mathrm{P}(H_1 \cup H_2 \cup H_3) = .1+.1+.1-.01-.01-.01+.001 = .271$ or $\mathrm{P}(H_1 \cup H_2 \cup H_3) = 1-\mathrm{P}(H_1^C H_2^C H_3^C) = 1-.9^3 = .271$

Problem 4. (20 points) Roll two fair dice. Let X be the largest value shown on the two dice.

(a) Write the probability mass function (pmf) of X.

(b) Write the cumulative density function (cdf) of X.

(c) Find $\mathrm{P}(4 \le X \le 6)$.

Solution:

(a) (6 points) The pmf of X is

x_i	$\mathrm{P}(X = x_i)$
1	1/36
2	3/36
3	5/36
4	7/36
5	9/36
6	11/36

(b) (8 points) The cdf of X is

$$F_X(a) = \begin{cases} 0, & \text{if } a < 1; \\ 1/36, & \text{if } 1 \le a < 2; \\ 4/36, & \text{if } 2 \le a < 3; \\ 9/36, & \text{if } 3 \le a < 4; \\ 16/36, & \text{if } 4 \le a < 5; \\ 25/36, & \text{if } 5 \le a < 6; \\ 1, & \text{if } a \ge 6. \end{cases}$$

(c) (6 points) $\mathrm{P}(4 \le X \le 6) = \mathrm{P}(X \le 6) - \mathrm{P}(X < 4) = F(6) - \lim_{n\to\infty} F(4-\frac{1}{n}) = 1 - 9/36 = 27/36 = 3/4$.

Bonus Problem. (3 points) A standard deck of cards has 52 cards with: 4 suits (hearts, diamonds, spades and clubs) and 13 cards in each suit (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen and king).

Cards are randomly drawn from a standard deck of 52 cards until the first ace appears. Given that the first ace is the $8th$ card drawn, what is the conditional probability that the tenth card drawn is the ace of spades? Do not simplify.

Solution:

Let A be the event that the first ace appears on the $8th$ draw.
Let S be the event that the $10th$ card drawn is the ace of spades.

$$\begin{aligned}
\mathrm{P}(A) &= \frac{\binom{48}{7}7!\binom{4}{1}}{\binom{52}{8}8!} \\
\mathrm{P}(AS) &= \frac{\binom{48}{7}\binom{3}{1}\binom{43}{1}7!}{\binom{52}{10}10!} \\
\mathrm{P}(S|A) &= \frac{\mathrm{P}(AS)}{\mathrm{P}(A)} \\
&= \frac{\frac{\binom{48}{7}\binom{3}{1}\binom{43}{1}7!}{\binom{52}{10}10!}}{\frac{\binom{48}{7}7!\binom{4}{1}}{\binom{52}{8}8!}} \\
&= \frac{3}{(4)(44)} \\
&= \frac{3}{176}
\end{aligned}$$

An equivalent way to find $\mathrm{P}(AS)$

$$\mathrm{P}(AS) = \frac{\binom{48}{8}\binom{3}{1}8! + \binom{48}{7}\binom{3}{1}\binom{2}{1}7!}{\binom{52}{10}10!}$$