APPM 3570 / STAT 3100
Fall 2021
Exam 3
December 13

- This exam has two parts, and you may start on either as long as you follow the instructions for each.
- Notes, your text and other books, cell phones, and other electronic devices are not permitted, except for calculators.
- Calculators are permitted.
- Write your name and sign and date your pledge to the CU Honor Code in the box below.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on either part of this work.
Name (Last, First): ______________________________
Signature: ______________________ **Date:** __________

ADDITIONAL INSTRUCTIONS FOR PART I: Use the last few pages of your bluebook as scratch paper to work out the following 15 questions. For each question, circle what you believe to be the correct answer. Ambiguous answers to a question will result in 0 points for that question. Do not justify your answers to these 15 questions.

1. (4 points.) If $X \sim \text{Exponential}(2)$ and $Y \sim \text{Normal}(0, 2)$ are independent, then $f(y|x)$ for $y \in \mathbb{R}$ and $x > 0$ is given by:

(a) $\frac{1}{2\sqrt{\pi}} e^{-y^2/4}$
(b) $\frac{1}{\sqrt{2\pi}} e^{-y^2/4}$
(c) $\frac{1}{4\pi} e^{-y^2/4}$
(d) $\frac{1}{2\pi} e^{-y^2/4}$
(e) $\frac{1}{2} e^{-2|y|}$
(f) None of the above

Solution: Since X and Y are independent, $f(y|x) = f_Y(y) = \frac{1}{\sqrt{4\pi}} e^{-y^2/4} = \frac{1}{2\sqrt{\pi}} e^{-y^2/4}$.

2. (4 points.) Consider a random vector (X, Y) such that $Y \sim \text{Uniform}(0,1)$ and the conditional distribution of X given that $Y = y$ is Normal with mean y^2 and variance 100. What's the expected value of X?

(a) 100
(b) 1/3
(c) 101
(d) 1/4
(e) 1/2
(f) None of the above

Solution: $E(X) = E\big(E(X|Y)\big) = E(Y^2) = \int_0^1 y^2 \cdot 1\, dy = 1/3.$

3. (4 points.) If X and Y are discrete random variables such that $p(x) = (1/2)^x$, for $x \geq 0$, and $p(y|x) = 1/(x+1)$, for $y = 0, \ldots, x$; what's the joint p.m.f. of X and Y?

(a) $p(x,y) = \begin{cases} \frac{2^x}{x+1} & \text{, for } 0 \leq y \leq x \text{ integers} \\ 0 & \text{, otherwise} \end{cases}$

(b) $p(x,y) = \begin{cases} (x+1)2^x & \text{, for } 0 \leq y \leq x \text{ integers} \\ 0 & \text{, otherwise} \end{cases}$

(c) $p(x,y) = \begin{cases} \frac{1}{(x+1)2^x} & \text{, for } 0 \leq y \leq x \text{ integers} \\ 0 & \text{, otherwise} \end{cases}$

(d) $p(x,y) = \begin{cases} \frac{x+1}{2^x} & \text{, for } 0 \leq y \leq x \text{ integers} \\ 0 & \text{, otherwise} \end{cases}$

(e) $p(x,y) = \begin{cases} \frac{1}{x+1} & \text{, for } 0 \leq y \leq x \text{ integers} \\ 0 & \text{, otherwise} \end{cases}$

(f) None of the above

Solution: (X, Y) can only take values of the form (x, y) with $0 \leq y \leq x$ integers, for which: $p(x,y) = P(X = x, Y = y) = P(X = x) \cdot P(Y = y|X = x) = (1/2)^x \cdot 1/(x+1)$, i.e. $p(x,y) = \begin{cases} \frac{1}{(x+1)2^x} & \text{, for } 0 \leq y \leq x \text{ integers;} \\ 0 & \text{, otherwise.} \end{cases}$

4. (4 points.) Suppose that $X \sim \text{Poisson}(\lambda)$, with $\lambda > 0$, and $P(Y = 0|X = k) = \frac{1}{k+1}$. What's the probability that $Y = 0$?

(a) λ
(b) $\frac{e^{\lambda}-1}{\lambda}$
(c) $\frac{1-e^{-\lambda}}{\lambda}$
(d) $\frac{\lambda}{2}$

(e) $\frac{1}{\lambda}$

(f) None of the above

Solution: $P(Y=0) = \sum_{k=0}^{\infty} P(Y=0|X=k) \cdot P(X=k) = \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} = \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} = \frac{e^{-\lambda}}{\lambda} \left(e^{\lambda} - 1\right) = \frac{1-e^{-\lambda}}{\lambda}$.

5. (4 points.) A grasshopper jumps one unit to the left with probability 1/3, and one unit to the right with probability 2/3. Call a sequence of two consecutive jumps a "waste" if the grasshopper returns to where it was two jumps ago. What's the expected number of wasted jumps in 10 jumps of the grasshopper?

(a) 4/9
(b) 40/9
(c) 9/2
(d) 5
(e) 4
(f) None of the above

Solution: Two consecutive jumps are a waste with probability $1/3 \cdot 2/3 + 2/3 \cdot 1/3 = 4/9$. In 10 jumps, the possible number of wasted jumps is 9, hence the expected number of wasted jumps is $9 \cdot 4/9 = 4$.

6. (4 points.) Imagine cereal boxes that contain one of four possible equally likely but different types of coupons. How many boxes should one expect to buy to collect all four coupon types for the first time?

(a) 4
(b) 5/2
(c) 40,000
(d) 25/3
(e) 22/3
(f) None of the above

Solution: For $1 \leq i \leq 4$, let X_i denote the random number of additional coupons one needs to buy to increase a collection containing $(i-1)$ different coupons to i. Since the coupons are equally likely, $X_i \sim \text{Geometric}((4-(i-1))/4)$. So, the expected number of boxes one should buy is $E(\sum_{i=1}^{4} X_i) = \sum_{i=1}^{4} E(X_i) = \sum_{i=1}^{4} \frac{4}{5-i} = \frac{4}{4} + \frac{4}{3} + \frac{4}{2} + \frac{4}{1} = \frac{25}{3}$.

7. (4 points.) If X and Y are random variables such that $\text{cov}(X,Y) = 0$, which of the following statements must be TRUE?

(a) X and Y are independent
(b) X and Y have the same distribution
(c) X and Y are discrete and have a joint p.m.f.
(d) X and Y are not independent
(e) X and Y are continuous and have a joint p.d.f.
(f) None of the above

Solution: None of the provided choices!

8. (4 points.) Let $n \geq 1$ be a very large integer, and $X_1, \ldots, X_n$ i.i.d. random variables with mean 2 and variance 25. Due to the *Law of Large Numbers* (LLN), $\frac{1}{n}\sum_{i=1}^n X_i^2$ should be with high probability approximately equal to:

(a) 29

(b) 2

(c) 5

(d) 25

(e) 4

(f) None of the above

Solution: Since $X_1^2, \ldots, X_n^2$ are also i.i.d., with $E(X_i^2) = V(X_i) + (EX_i)^2 = 25 + 2^2 = 29$, the LLN implies that $\frac{1}{n}\sum_{i=1}^n X_i^2 \approx 29$.

9. (4 points.) Let X and Y be continuous random variables with joint p.d.f. $f(x,y) = (2e-5)\, x\, e^y\, /2$ for $x, y \geq 0$ such that $x + y \leq 1$, otherwise $f(x,y) = 0$. What's the conditional p.d.f. of X given that $Y = 1/2$?

(a) $8x$, for $0 \leq x \leq 1$, otherwise it vanishes

(b) x, for $0 \leq x \leq 1/2$, otherwise it vanishes

(c) $8x$, for $0 \leq x \leq 1/2$, otherwise it vanishes

(d) $(2e-5)\, x\, e^{1/2}\, /2$, for $0 \leq x \leq 1/2$, otherwise it vanishes

(e) $2x$, for $0 \leq x \leq 1$, otherwise it vanishes

(f) None of the above

Solution: $f(x|y = 1/2) = \frac{x \cdot e^y}{f_Y(y=1/2)} = c \cdot x$, for $0 \leq x \leq 1/2$, where c is a suitable constant such that $\int_0^{1/2} f(x|y = 1/2)\, dx = 1$. So $c = 8$, i.e. $f(x|y = 1/2) = 8x$ for $0 \leq x \leq 1/2$, otherwise $f(x|y = 1/2) = 0$.

10. (4 points.) Let X, Y, Z be random variables such that $\mathrm{cov}(X,Y) = 1$, $\mathrm{cov}(X,Z) = -2$, $\mathrm{cov}(Y,Z) = 3$, and $V(Z) = 5$. What's $\mathrm{cov}(Y+Z, X-Z)$?

(a) 1

(b) -9

(c) -3

(d) 7

(e) -11

(f) None of the above

Solution: $\mathrm{cov}(Y+Z, X-Z) = \mathrm{cov}(Y,X) - \mathrm{cov}(Y,Z) + \mathrm{cov}(Z,X) - \mathrm{cov}(Z,Z) = 1 - 3 + (-2) - V(Z) = -4 - V(Z) = -9$.

11. (4 points.) If X, Y, and Z are independent random variables taking values in $\mathbb{Z}$ (i.e., only integer values), what's the probability that $X + Y = 9$ and $Y - Z = 5$?

(a) $\sum_{k\in\mathbb{Z}} p_X(9+k)\cdot p_Y(k)\cdot p_Z(k+5)$

(b) $\sum_{k\in\mathbb{Z}} p_X(9-k)\cdot p_Y(k)\cdot p_Z(k-5)$

(c) $\sum_{k\in\mathbb{Z}} p_X(9-k)\cdot p_Y(k)\cdot p_Z(k+5)$

(d) $\sum_{k\in\mathbb{Z}} p_X(9+k)\cdot p_Y(k)\cdot p_Z(k-5)$

(e) $p_X(9)\cdot p_Y(0)\cdot p_Z(-5)$

(f) None of the above

Solution:

$$\begin{aligned}P(X+Y=9, Y-Z=5) &= \sum_{k\in\mathbb{Z}} P(X+Y=9, Y-Z=5 \mid Y=k)\cdot P(Y=k)\\ &= \sum_{k\in\mathbb{Z}} P(X+k=9, k-Z=5 \mid Y=k)\cdot p_Y(k)\\ &= \sum_{k\in\mathbb{Z}} P(X=9-k, Z=k-5)\cdot p_Y(k)\\ &= \sum_{k\in\mathbb{Z}} p_X(9-k)\cdot p_Y(k)\cdot p_Z(k-5).\end{aligned}$$

12. (4 points.) The average weight of a carry-on bag in commercial flights is 12 lbs, with a standard deviation of 4. What's approximately the probability that the carry-on bags of 100 passengers will exceed 1400 lbs? Leave your answer in terms of Φ, the c.d.f. of a standard Normal.

(a) $\Phi(200)$

(b) $1-\Phi(5)$

(c) $\Phi(5)$

(d) $1-\Phi(200)$

(e) $1-\Phi(1400)$

(f) None of the above

Solution: Let W_i the weight of the carry-on bag of the i-th passenger. In particular, $E(\sum_{i=1}^{100} W_i) = 1200$ and $SD(\sum_{i=1}^{100} W_i) = \sqrt{100\cdot 4^2} = 40$. So, due to the CLT: $P(\sum_{i=1}^{100} W_i > 1400) = P(\frac{\sum_{i=1}^{100} W_i - 1200}{40} > \frac{1400-1200}{40}) \approx P(\text{Standard Normal} > 5) = 1-\Phi(5)$.

13. (4 points.) If X, Y, and Z are dependent random variables such that $E(X) = 4$, $E(Y) = 7$, and $E(Z) = 4$, what's the expected value of $X - 2Y + Z/2$?

(a) 20
(b) -12
(c) -16
(d) -8
(e) It cannot be computed with the information given!
(f) None of the above

Solution: $E(X - 2Y + Z/2) = E(X) - 2E(Y) + E(Z)/2 = 4 - 14 + 2 = -8.$

14. (4 points.) If X is a random variable with variance 3, what's the variance of $1-2X$?

(a) 6
(b) 12
(c) -6
(d) -12
(e) 5
(f) None of the above

Solution: $V(1-2X) = V(-2X) = 4 \cdot V(X) = 12.$

15. (4 points.) Let $X_1, \ldots, X_m$ be Binomial random variables with parameters $(n, 1/2)$. If for $i \neq j$ the covariance between X_i and X_j is (-1), what's the variance of $\frac{1}{\sqrt{m}} \sum_{i=1}^{m} X_i$?

(a) $\frac{n}{4}$
(b) $\frac{n}{2} - m + 1$
(c) $\frac{n}{4} - m + 1$
(d) $\frac{n}{4} + m - 1$
(e) $\frac{n}{2} + m - 1$
(f) None of the above

Solution: $V(\frac{1}{\sqrt{m}} \sum_{i=1}^{m} X_i) = \frac{1}{m} V(\sum_{i=1}^{m} X_i) = \frac{1}{m} \sum_{i=1}^{m} V(X_i) + \frac{1}{m} \sum_{i \neq j}^{m} \text{cov}(X_i, X_j) = \frac{1}{m} \sum_{i=1}^{m} \frac{n}{4} + \frac{1}{m} \sum_{i \neq j}^{m} (-1) = \frac{n}{4} - (m-1) = \frac{n}{4} - m + 1.$ rblack

*** One More Questions Ahead! ***

INSTRUCTIONS FOR PART II: Use the first pages of your bluebook to answer to the following question. On the front cover of your bluebook, write (i) your name; and (ii) when your class meets (9 AM, or 3 PM). Do all parts of the problem. Box in your answers, and make sure to **show all your work and justify your answers**.

Please draw a grading table with three rows and two columns on the front of your bluebook.

Problem A. (40 points.) Let X and Y be continuous random variables with joint p.d.f. given by the formula:

$$f(x,y) = \begin{cases} \frac{\exp(-x^2/2)}{3} & , x > 0 \text{ and } -x < y < 2x; \\ 0 & , \text{otherwise.} \end{cases}$$

For $x > 0$, respond:

(a) What's the p.d.f. $f(x)$ of X? If this distribution is known, name it and identify its parameters.

(b) What's the conditional p.d.f. $f(y|x)$? If this distribution is known, name it and identify its parameters.

(c) What's the conditional expected value $E(Y|X = x)$?

(d) Determine $E(Y)$ explicitly!

The next two questions are unrelated from the previous ones.

(e) Determine the joint p.d.f. of $U := X$ and $V := XY$.

(f) Are U and V independent? Explain!

Solution:

(a) $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-x}^{2x} \frac{e^{-x^2/2}}{3}\,dy = x\,e^{-x^2/2}$.

(b) For $-x < y < 2x$: $f(y|x) = \frac{f(x,y)}{f(x)} = \frac{e^{-x^2/2}/3}{xe^{-x^2/2}} = \frac{1}{3x}$, otherwise $f(y|x) = 0$. So, the conditional distribution of Y given that $X = x$ is Uniform$(-x, 2x)$.

(c) From part (b), the conditional expected value of Y given that $X = x$ is the mid-point of the interval $(-x, 2x)$, i.e. $E(Y|X = x) = \frac{-x+2x}{2} = \frac{x}{2}$.

(d) **Solution I.**
$E(Y) = E\big(E(Y|X)\big) = \int_{-\infty}^{\infty} E(Y|X = x)\cdot f(x)\,dx = \int_0^{\infty} \frac{x^2}{2}\cdot e^{-x^2/2}\,dx = \int_0^{\infty} \frac{x}{2}\cdot\left(-e^{-x^2/2}\right)'\,dx = \frac{1}{2}\int_0^{\infty} e^{-x^2/2}\,dx = \frac{\sqrt{2\pi}}{2}\int_0^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \frac{\sqrt{2\pi}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2\pi}}{4}$.

Solution II.
$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\cdot f(x,y)\,dy\,dx = \int_0^{\infty}\int_{-x}^{2x} y\cdot\frac{e^{-x^2/2}}{3}\,dy\,dx = \int_0^{\infty} \frac{x^2}{2}\cdot e^{-x^2/2}\,dx = \ldots = \frac{\sqrt{2\pi}}{4}$.

(e) Observe that if $x > 0$ and $-x < y < 2x$, and $u := x$ and $v := xy$, then $u > 0$ and $-u^2 < v < 2u^2$. For (u, v) satisfying these constraints

$$f(u,v) = \frac{f(x,y)}{\left|\det\begin{pmatrix}1 & 0\\ y & x\end{pmatrix}\right|} = \frac{e^{-x^2/2}/3}{x} = \frac{e^{-x^2/2}}{3x} = \frac{e^{-u^2/2}}{3u}.$$

In other words:

$$f(u,v) = \begin{cases}\frac{e^{-u^2/2}}{3u} & , u > 0 \text{ and } -u^2 < v < 2u^2;\\ 0 & , \text{ otherwise.}\end{cases}$$

(f) U and V are not independent because in principle V can take any real value, however, $-U^2 < V < 2U^2$ i.e. U restricts V.

Duration: 90 minutes