APPM 3570 / STAT 3100
Fall 2021
Exam 2
November 10

- This exam has two parts, and you may start on either as long as you follow the instructions for each.
- Notes, your text and other books, cell phones, and other electronic devices are not permitted, except for calculators.
- Calculators are permitted.
- Write your name and sign and date your pledge to the CU Honor Code in the box below.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on either part of this work.
Name (Last, First): ______________________________
Signature: ______________________ **Date:** ____________

ADDITIONAL INSTRUCTIONS FOR PART I: Use the last few pages of your bluebook as scratch paper to work out the following 10 questions. For each question, circle what you believe to be the correct answer. Ambiguous answers to a question will result in 0 points for that question. Do not justify your answers to these 10 questions.

1. (4 points.) Let X and Y be discrete random variables with joint p.m.f. given by the following table:

	Y=0	Y=1
X=0	1/12	1/4
X=1	1/6	1/2

Which of the following statements is FALSE?

Solution: $P(X=1)=1/6+1/2=4/6=2/3$; $P(Y=0)=1/12+1/6=3/12=1/4$; $P(X\cdot Y=0)=1-P(X=1,Y=1)=1-1/2=1/2$; X and Y are independent because $P(X=0,Y=0)=1/12=(1-2/3)\cdot 1/4=P(X=0)\cdot P(Y=0)$, $P(X=0,Y=1)=1/4=(1-2/3)\cdot(1-1/4)=P(X=0)\cdot P(Y=1)$, $P(X=1,Y=0)=1/6=2/3\cdot 1/4=P(X=1)\cdot P(Y=0)$, $P(X=1,Y=1)=1/2=2/3\cdot(1-1/4)=P(X=1)\cdot P(Y=1)$; $P(X+Y=1)=P(X=0,Y=1)+P(X=1,Y=0)=1/4+1/6=5/12$. So, $P(X+Y=1)=1/24$ is FALSE.

2. (4 points.) Suppose you play a game with a biased coin, which has a probability of observing heads equal to 2/3. The probability of observing tails is 1/3. You win \$10 if the coin lands on tails, and you lose \$6 if the coin lands on heads. What's the variance of the dollars won? ("Dollars won" can be negative.)

Solution: $E[X^2] = \frac{172}{3}$ or 57.333, $E[X] = -\frac{2}{3}$, so $Var(X) = \frac{512}{9}$ or 56.889.

3. (4 points.) If $X \sim Binomial(n = 100, p = 1/5)$, which of the following statements is FALSE?

Solution: $E(X) = np = 20$; $V(X) = np(1-p) = 16$; $(X-20)/4 = (X - E(X))/\sqrt{V(X)}$ has approximately a standard Normal distribution; $P(X = 50) = \binom{n}{50}p^{50}(1-p)^{n-50} = \binom{100}{50}\frac{4^{50}}{5^{100}}$; and $P(X = 0) = \binom{n}{0}p^0(1-p)^{n-0} = \left(\frac{4}{5}\right)^{100} \neq \frac{1}{5^{100}}$.

4. (4 points.) Suppose the number of hits a web site receives in any time interval is a Poisson random variable. If a particular site gets on average 5 hits per second, what's the probability it will get no hits in an interval of two seconds?

Solution: $X \sim Poisson(10), P(X = 0) = e^{-10}$.

5. (4 points.) The average number of acres burned by forest and range fires in a Colorado county is 700 acres per year, with a standard deviation of 360 acres. If the number of acres burned is Normal distributed, what's the probability that between 520 and 970 acres will be burned in any given year?
Answer in terms of Φ, the c.d.f. of a standard Normal distribution, evaluated at non-negative numbers.

Solution: Let X be the number of acres burned on a given year. Then $P(520 \leq X \leq 970) = P\left(\frac{520-700}{360} \leq \frac{X-700}{360} \leq \frac{970-700}{360}\right) = P\left(\frac{-1}{2} \leq \frac{X-700}{360} \leq \frac{3}{4}\right) = \Phi(3/4) - \Phi(-1/2) = \Phi(3/4) + \Phi(1/2) - 1$.

6. (4 points.) The time in hours required to repair a machine is an exponentially distributed random variable with rate parameter $\lambda = 1/4$. What's the probability that a repair time exceeds 4 hours?

Solution: $P(X > 4) = \int_4^\infty \frac{1}{4}e^{-x/4}dx = e^{-1}$.

7. (4 points.) If U is a Uniform r.v. on the interval $[-1, 2]$, then the expected value of $U \cdot (U - 1)$ is:

Solution: $E\big(U(U-1)\big) = \int_{-1}^2 u(u-1) \cdot \frac{1}{3}\, du = \frac{1}{3}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Big|_{u=-1}^{u=2} = \frac{1}{2}$.

8. (4 points.) If $X \sim \text{Uniform}(0, 5)$, what's the p.d.f. of $Y = e^X$?

Solution: Clearly, $1 < Y < e^5$, and $f_Y(y) = 0$ for $y \leq 1$ and for $y \geq e^5$. For $1 < y < e^5$: $f_Y(y) = f_X(g^{-1}(y)|\frac{d}{dy}g^{-1}(y)| = \frac{1}{5}|\frac{d}{dy}g^{-1}(y)| = \frac{1}{5}|\frac{d}{dy}\ln y| = \frac{1}{5y}$.

9. (4 points.) Let X and Y be random variables with joint p.d.f.

$$f(x,y) = \begin{cases} 2\,e^{-x-y} & , x > 0 \text{ and } \frac{x}{2} < y < 5x; \\ 0 & , \text{otherwise.} \end{cases}$$

What's the marginal p.d.f. of Y for $y > 0$?

Solution: For $y > 0$: $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{y/5}^{2y} 2\,e^{-x-y}\,dx = 2\,e^{-y}\,(-e^{-x})\big|_{x=y/5}^{x=2y} = 2\left(e^{-6y/5} - e^{-3y}\right)$.

10. (4 points.) The joint p.m.f. of the random variables X, Y, Z is:

$$P(X = x, Y = y, Z = z) = \frac{1}{4}, \text{ if } (x, y, z) = (1, 2, 3), (2, 1, 1), (2, 2, 1), \text{ or } (2, 3, 2).$$

What's the conditional probability $P(XYZ = 2 \,|\, Z = 1)$?

Solution: $P(XYZ = 2 \,|\, Z = 1) = \frac{P(XYZ=2,\, Z=1)}{P(Z=1)} = \frac{1/4}{1/2} = \frac{1}{2}$.

*** Two More Questions Ahead! ***

INSTRUCTIONS FOR PART II: Use the front of your bluebook to answer to the following 2 questions. On the front cover of your bluebook, write (i) your name; and (ii) when your class meets (9 AM, or 3 PM). Do all parts of each problem. Start each problem on a new page of your bluebook. $\boxed{\text{Box}}$ in your answers, and make sure to **show all your work and justify your answers**. Also, please draw a grading table with four rows and two columns on the front of your bluebook.

Problem A. (30 points.) Suppose the joint p.d.f. of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 15xy^2 & , 0 < y < x < c; \\ 0 & , \text{ otherwise.} \end{cases}$$

(a) Find the value of the constant c so that the joint p.d.f. is valid.

(b) Calculate $P(2Y - X > 0)$.

(c) Are X and Y independent? Justify!

Solution:

(a) Note that $f_{X,Y} \geq 0$.

$$\int_0^c \int_0^x 15xy^2\, dy\, dx = \int_0^c 5x^4\, dx = c^5.$$

Therefore need $c = 1$.

(b)

$$\begin{aligned} P(2Y - X > 0) &= P(Y > \frac{X}{2}) \\ &= \int_0^1 \int_{\frac{x}{2}}^x 15xy^2\, dy\, dx \\ &= \int_0^1 5x(y^3\Big|_{\frac{x}{2}}^x)\, dx \\ &= \int_0^1 (5x^4 - \frac{5}{8}x^4)\, dx \\ &= (x^5 - \frac{1}{8}x^5)\Big|_0^1 = \frac{7}{8} \end{aligned}$$

(c) **Solution I.** For $0 < x < 1$:

$$f_X(x) = \int_0^x 15xy^2\, dy = 5x^4.$$

For $0 < y < 1$:

$$f_Y(y) = \int_y^1 15xy^2\, dx = \frac{15y^2(1-y^2)}{2}.$$

So $f(x,y) = 15xy^2 \neq 5x^4 \cdot \frac{15y^2(1-y^2)}{2} = f_X(x) \cdot f_Y(y)$, for $x, y \in (0,1)$, hence X and Y cannot be independent.

Solution II. In principle, X may take any value between 0 and 1. However, from the joint p.d.f. if $Y = y$ then $y < X$, i.e., knowledge of Y affects how X behaves, hence X and Y cannot be independent. (A similar argument follows from observing how X affects Y.)

Problem B. (30 points.) Let X and Y be discrete random variables with joint probability mass function (p.m.f.):

$$p(x,y) = \frac{1}{e \cdot (x+1)!}\left(1 - \frac{1}{x+1}\right)^{y-1}, \text{ for integers } x \geq 0 \text{ and } y \geq 1.$$

(a) Show that X has a Poisson distribution with parameter $\lambda = 1$.
Hint. $\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}$, when $|\rho| < 1$.

(b) For each integer $x \geq 0$, what's the conditional distribution of Y given that $X = x$? If this distribution is known, name it and identify its parameters.

(c) Are X and Y independent? Explain!

Solution:

(a) For a fixed integer $x \geq 0$:

$$\begin{aligned}
p_X(x) &= \sum_{y=1}^{\infty} p(x,y) \\
&= \frac{1}{e \cdot (x+1)!} \sum_{y=1}^{\infty} \left(1 - \frac{1}{x+1}\right)^{y-1} \\
&= \frac{1}{e \cdot (x+1)!} \cdot \frac{1}{1 - \left(1 - \frac{1}{x+1}\right)} \\
&= \frac{x+1}{e \cdot (x+1)!} \\
&= \frac{1}{e \cdot x!} = \frac{1^x}{x!} e^{-1},
\end{aligned}$$

which is the p.m.f. of a $Poisson(\lambda = 1)$.

(b) For integers $x \geq 0$ and $y \geq 1$:

$$p(y|x) = \frac{p(x,y)}{p_X(x)} \overset{(a)}{=} \frac{\frac{1}{e \cdot (x+1)!}\left(1 - \frac{1}{x+1}\right)^{y-1}}{\frac{1}{e \cdot x!}} = \left(1 - \frac{1}{x+1}\right)^{y-1} \frac{1}{x+1},$$

which is the p.m.f. of a Geometric with parameter $p = 1/(x+1)$.

(c) **Solution I.** X and Y are not independent because, from part (b), the conditional distribution of Y given that $X = x$ depends on x, which could not be the case if X and Y were independent.

Solution II.

$$P(X=1, Y=1) = p(1,1) = \frac{1}{2e};$$

$$P(X=1) = \frac{1}{e};$$

$$P(Y=1) = \sum_{x=0}^{\infty} p(x,1) = \frac{1}{e}\sum_{x=0}^{\infty}\frac{1}{(x+1)!} = \frac{1}{e}\left\{\sum_{k=0}^{\infty}\frac{1}{k!} - \frac{1}{0!}\right\} = \frac{e-1}{e}.$$

Since

$$\frac{1}{2e} \neq \frac{1}{e}\cdot\frac{e-1}{e},$$

X and Y are not independent.